

On the regularity of fractional integrals of modular forms.

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Abstract

In this paper we study some local and global regularity properties of Fourier series obtained as fractional integrals of modular forms. In particular we characterize the differentiability at rational points, determine their Hölder exponent everywhere (using several definitions) and compute the associated spectrum of singularities. We also show that these functions satisfy an approximate functional equation, and use it to discuss the graphs of “Riemann’s example” and of fractional integrals of cusp forms for $\Gamma_0(N)$. We include some computer plots.

1 Introduction

The function

$$\varphi(x) = \sum_{n \geq 1} \frac{\sin(n^2 \pi x)}{n^2} \quad (1)$$

was introduced in [24] by Weierstrass as an example supposedly given by Riemann of a continuous function which is not differentiable anywhere. It was later verified by Hardy [13] that this is indeed the case except perhaps at the rational numbers of the form odd/odd or even/(4n + 1). The behavior at the remaining points was not known until 1970, when Gerver proved in [11, 12] that φ is actually differentiable at those rationals of the form odd/odd while is not in the other case (these assertions are almost evident from its graph when plotted with the aid of modern computers; see figure 1).

In the light of historical analysis [4] it seems probable that Riemann never made such a claim. In spite of this, φ has become known in the literature as “Riemann’s example”, and its regularity has been extensively studied by several authors. What lies underneath its apparently chaotic behavior is the action of a certain subgroup of $\mathrm{SL}_2(\mathbb{Z})$ on Jacobi’s theta function

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{n^2 \pi i z} \quad (\Im(z) > 0),$$

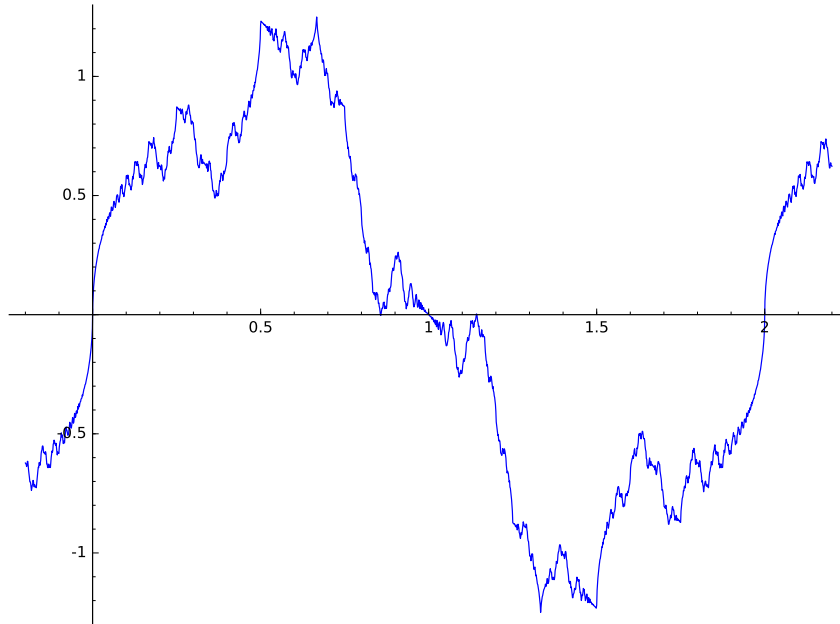


Figure 1: Riemann's example.

which is a modular form of weight $1/2$. The connection can be formally stated as $2\pi^{-1}\varphi'(x) = \Re\theta(x) - 1$. This leads to very fruitful strategies that can be used to study φ : for example, in [14] and [17] the derivative is understood as a certain wavelet transform and general theorems are applied which relate bounds on one side of the transform with regularity on the other; while in [7] an approximate functional equation for φ is deduced integrating the one for θ .

In the same spirit, if we are given any modular form of positive weight for a subgroup of finite index of $SL_2(\mathbb{Z})$ one can perform a formal fractional integration on its Fourier series to force it to converge to a continuous function on all the real line. The regularity of the resulting function (understood in several ways) has been studied for certain limited ranges in [5] and more recently [6]. In this last paper, for example, Chamizo *et al.* determine under some restrictions the so called pointwise Hölder exponent, which measures how well a function can be locally approximated by polynomials.

In this paper we intertwine the wavelet transform and approximate functional equation approaches in order to study the general case. Indeed neither technique suffices to determine the pointwise Hölder exponent everywhere,

and one needs to resort to a combination of them. We also determine unconditionally other related local exponents, and characterize the rational points at which the fractional integral is differentiable. Moreover we compute the spectrum of singularities, which consists of the Hausdorff dimension of the sets where the function attains a certain pointwise Hölder exponent. For the precise definitions see §2.

As noted by Duistermaat in [7], the approximate functional equation deepens our understanding of the graph of these functions. In particular it shows that around rational numbers we should expect oscillations of the form $x^a g(1/x)$, where g is a periodic function also given by a fractional integral of a modular form (these oscillations are clearly visible in figure 1). Another consequence of the functional equation is that these graphs are fractal in the sense that they satisfy a very particular kind of approximate self-similarity around rational numbers and quadratic surds. These features are studied in detail in the case of “Riemann’s example”, and all the possibilities for g are determined.

We also perform a brief analysis of the case of cusp forms for the group $\Gamma_0(N)$, that is, the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of those matrices whose bottom-left entry is divisible by N . In this setting the vector space of cusp forms has a basis consisting of normalized eigenfunctions for all the Hecke operators, and the elements of this basis are classified into newforms and oldforms, depending on whether they originate on $\Gamma_0(N)$ or derive from forms on $\Gamma_0(N_1)$ for some proper divisor N_1 of N (see [2]). The modularity theorem (one of the major mathematical achievements of contemporary mathematics) states that there is a correspondence between isogeny classes of elliptic curves over \mathbb{Q} of conductor N and newforms of weight 2 for the group $\Gamma_0(N)$ and trivial multiplier system. The coefficients of these newforms coincide with those of the Hasse-Weil L-function of any elliptic curve in the associated isogeny class.

It is known that some arithmetic properties of the curve are encoded into analytic properties of the L-function: the Birch and Swinnerton-Dyer conjecture—of which some particular cases have been proven—relates the rank of the group of rational points of the curve with the order of the zero at $z = 1$ of the L-function. In [5] it is shown that some of these analytic properties are also reflected on fractional integrals of the corresponding newforms, and can be read off their graphs.

Here we focus on a different aspect: we study how the action of the normalizer of $\Gamma_0(N)$ on \mathbb{Q} and on the newform constraints the possible set of functions g for which we should expect oscillations of type $x^a g(1/x)$ around rational numbers. More precisely, we characterize when the normalizer acts

transitively on \mathbb{Q} and then prove that if some extra conditions are met we can assure g to be, again, a fractional integral of the original newform.

The layout of this paper is the following: in §2 we state our main results. §3 contains some preliminary lemmas, while §4 and §5 are devoted to developing the tools concerning the approximate functional equation and wavelet approaches. In §6 we focus on determining the Hölder exponents while §7 deals with the spectrum of singularities. Finally, §8 and §9 correspond to the φ and $\Gamma_0(N)$ cases, and include computer-generated images of both of them.

2 Notation and main results

The notation $f(x) \ll g(x)$, or $f = O(g)$, will be employed to denote that the inequality $|f(x)| \leq C|g(x)|$ is satisfied for some unspecified positive constant C , usually as x converges to a certain point.

We introduce the following spaces of complex-valued functions, defined either in all \mathbb{R} or in an open subset of \mathbb{R} .

- For $0 \leq s \leq 1$ we define $\Lambda^s(x_0)$ as the set of all continuous functions which satisfy a s -Hölder condition at x_0 , *i.e.*,

$$|f(x) - f(x_0)| \ll |x - x_0|^s \quad (x \rightarrow x_0),$$

and analogously one defines $\Lambda^s(\Omega)$ for any subset $\Omega \subset \mathbb{R}$ as the set of all continuous functions satisfying a uniform s -Hölder condition on Ω .

- For any $s \geq 0$ we define $\mathcal{C}^s(x_0)$ as the set of all continuous functions for which there is some polynomial P of degree less than or equal to s satisfying

$$|f(x) - P(x - x_0)| \ll |x - x_0|^s \quad (x \rightarrow x_0).$$

- For any $0 \leq s \leq 1$ and any integer $k \geq 0$ we define $\mathcal{C}^{k,s}(x_0)$ as the set of all continuous functions for which $f^{(k)}$ exists in an interval I containing x_0 and verifies $f^{(k)} \in \Lambda^s(x_0)$. Analogously one defines $\mathcal{C}^{k,s}(\Omega)$ for an open set $\Omega \subset \mathbb{R}$ as the set of all continuous functions for which $f^{(k)}$ exists in Ω and $f^{(k)} \in \Lambda^s(K)$ for every compact subset $K \subset \Omega$.

Finally one defines the spaces Λ_{\log}^s , \mathcal{C}_{\log}^s and $\mathcal{C}_{\log}^{k,s}$ by replacing $|x - x_0|^s$ in the previous definitions by $|x - x_0|^s \log |x - x_0|$.

We choose the following Hölder exponents as measures of the local regularity of a function f at a certain point (*cf.* [6, 22]):

$$\begin{aligned}\beta(x_0) &:= \sup\{s : f \in \mathcal{C}^s(x_0)\}, \\ \beta^*(x_0) &:= \sup\{k + s : f \in \mathcal{C}^{k,s}(x_0)\}, \\ \beta^{**}(x_0) &:= \lim_{I \rightarrow \{x_0\}} \sup\{k + s : f \in \mathcal{C}^{k,s}(I)\}.\end{aligned}$$

The first one, $\beta(x_0)$, also called the pointwise Hölder exponent, is the most local in nature and gives precise information about how well the function can be approximated by a polynomial in arbitrarily small neighborhoods of x_0 , even when no derivative exists near that point (note that P in the definition of $\mathcal{C}^s(x_0)$ need not be the Taylor polynomial).

The second one, $\beta^*(x_0)$, also called the restricted local Hölder exponent, is more demanding in the sense that f must be differentiable enough times for it to coincide with $\beta(x_0)$. This is in some sense like imposing that the polynomial is the actual Taylor polynomial of f .

Finally, $\beta^{**}(x_0)$, the local Hölder exponent, requires f not only to be differentiable in open neighborhoods, but also its derivatives to satisfy a Hölder condition in them. The importance of this last one resides in the fact that it behaves well under the action of a wide class of pseudo-differential operators (*cf.* [22]).

It is not hard to prove that they satisfy the inequalities

$$\beta(x) \geq \beta^*(x) \geq \beta^{**}(x),$$

and that there are examples for which both inequalities are strict (it suffices to consider functions of the form $x^a \sin x^{-b}$; see [6]).

Unless otherwise stated from now on f will denote a nonzero modular form of weight r for a subgroup Γ of finite index of $\mathrm{SL}_2(\mathbb{Z})$ and multiplier system $\{\mu_\gamma\}$ (*cf.* [16, 21]). This means that f is analytic in the upper half-plane \mathbb{H} , transforms by the law

$$f(\gamma z) = \mu_\gamma(cz + d)^r f(z) \quad \text{for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad (2)$$

where $|\mu_\gamma| = 1$, and has at most polynomial growth when $\Im z \rightarrow 0^+$. The term γz stands for the Möbius transformation $(az + b)/(cz + d)$. We also introduce the classical notation $j_\gamma(z) := cz + d$. All the power and logarithm functions considered in this article correspond to the branch with argument determination $-\pi < \arg z \leq \pi$.

Given any matrix γ in $\mathrm{GL}_2^+(\mathbb{R})$ we define the slash operator $|_\gamma$ acting on f by

$$f|_\gamma(z) := (\det \gamma)^{r/2} \frac{f(\gamma z)}{(j_\gamma(z))^r}.$$

In particular if $\gamma \in \Gamma$ we have $f|_\gamma = \mu_\gamma f$. More generally if the group $\gamma^{-1}\Gamma\gamma \cap \mathrm{SL}_2(\mathbb{Z})$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$ then $f|_\gamma$ is a modular form of weight r for this group in the sense defined above, albeit the multiplier system might change even if $\gamma^{-1}\Gamma\gamma = \Gamma$. The finiteness condition is satisfied in particular for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. We also note that the slash operator satisfies $(f|_\gamma)|_\delta = f|_{\gamma\delta}$ for any two matrices γ and δ .

We will employ the nonstandard notation f^γ to mean the same as $f|_\gamma$ to avoid complications when adding subscripts.

For a cusp we mean either a rational number or ∞ . Under the previous conditions f admits a “Fourier” expansion at every cusp \mathfrak{a} (cf. [16])

$$f^{\sigma_{\mathfrak{a}}}(z) = \sum_{n \geq 0} a_n^{\mathfrak{a}} e^{2\pi i(n + \kappa_{\mathfrak{a}})z} \quad (z \in \mathbb{H}). \quad (3)$$

In this expression $\sigma_{\mathfrak{a}}$ stands for a scaling matrix for the cusp \mathfrak{a} , which corresponds to a product $\gamma\eta$ where γ is any matrix in $\mathrm{SL}_2(\mathbb{Z})$ satisfying $\gamma(\infty) = \mathfrak{a}$ and $\eta = \begin{pmatrix} \sqrt{m_{\mathfrak{a}}} & 0 \\ 0 & 1/\sqrt{m_{\mathfrak{a}}} \end{pmatrix}$, $m_{\mathfrak{a}}$ denoting the width of \mathfrak{a} . To avoid the ambiguity of choosing between γ and $-\gamma$ we will adopt the convention that either $c < 0$, or $c = 0$ and $d = 1$, where (c, d) is the bottom row of γ . The cusp parameter $0 \leq \kappa_{\mathfrak{a}} < 1$ depends only on \mathfrak{a} , while the coefficients $a_n^{\mathfrak{a}} \in \mathbb{C}$ may assume a finite number of values, as multiplication of γ on the right by an unit translation corresponds to the change of variables $z \mapsto z + 1/m_{\mathfrak{a}}$. Up to this, they are unique. Moreover if one replaces \mathfrak{a} with any other cusp lying in the same orbit modulo Γ , the right hand side of (3) stays invariant up to multiplication by an unimodular constant and the aforementioned translations.

If either $\kappa_{\mathfrak{a}} > 0$ or $a_0^{\mathfrak{a}} = 0$ then we say that f is cuspidal at \mathfrak{a} , or that \mathfrak{a} is cuspidal for f , and we define $f(\mathfrak{a}) = 0$. Otherwise we define $f(\mathfrak{a}) = a_0^{\mathfrak{a}}$. Finally, if f is cuspidal at every cusp we say that f is a cusp form.

We will assume that $\kappa_{\mathfrak{a}} \in \mathbb{Q}$ for any cusp \mathfrak{a} . This is not a big restriction since any modular form coming from an arithmetic setting satisfies this, and most examples are of this kind. Notice that under this assumption (3) is, up to a dilation, a Fourier series in the usual sense.

In the case $\sigma_{\infty} = \eta$ expression (3) corresponds to

$$f(mz) = \sum_{n \geq 0} a_n e^{2\pi i(n + \kappa)z} \quad (z \in \mathbb{H}), \quad (4)$$

where $a_n = a_n^\infty$, $m = m_\infty$, $\kappa = \kappa_\infty$. Given $\alpha > 0$ we define the α -fractional integral of f as the formal series (cf. [5, 6, 7, 17])

$$f_\alpha(mx) := \sum_{n+\kappa>0} \frac{a_n}{(n+\kappa)^\alpha} e^{2\pi i(n+\kappa)x}. \quad (5)$$

For example, with the notation used in the introduction, $\Im\theta_1(x) = 2\varphi(x)$.

The group $\sigma_a^{-1}\Gamma\sigma_a \cap \mathrm{SL}_2(\mathbb{Z})$ always has finite index in $\mathrm{SL}_2(\mathbb{Z})$ and therefore $f_\alpha^{\sigma_a} := (f^{\sigma_a})_\alpha$ makes sense. In this fashion we obtain a collection of formal series

$$f_\alpha^{\sigma_a}(x) = \sum_{n+\kappa_a>0} \frac{a_n^a}{(n+\kappa_a)^\alpha} e^{2\pi i(n+\kappa_a)x}.$$

Note that $f_\alpha^{\sigma_a}$ is uniquely determined by the orbit of the cusp \mathfrak{a} up to translation and multiplication by an unimodular constant. As we will see, these $f_\alpha^{\sigma_a}$ are intimately linked one to each other.

Although the results in this section are stated for an arbitrary modular form, in the proofs (§§3-6) we will restrict to the case $m_\infty = 1$, $\kappa_\infty = 0$. This simplifies some arguments and can be assumed without loss of generality, as up to dilation any modular form is of this kind.

Our first three theorems establish some global and local regularity properties of f_α . We use the following notation: for any real s we denote by $[s]$ its integer part, *i.e.*, the biggest integer which is smaller than or equal to s , and by $\{s\}$ its decimal part $s - [s]$. We also define $\alpha_0 := r/2$ if f is a cusp form and $\alpha_0 := r$ otherwise.

Theorem 1 (Global regularity). *The following holds:*

1. For $\alpha \leq \alpha_0$ the series (5) defining f_α diverges in a dense set.
2. For $\alpha > \alpha_0$ the series (5) defining f_α converges uniformly to a continuous function in all the real line. Moreover $f_\alpha \in C^{[\alpha-\alpha_0], \{\alpha-\alpha_0\}}(\mathbb{R})$ if $\alpha - \alpha_0 \notin \mathbb{Z}$ and $f_\alpha \in C_{\log}^{\alpha-\alpha_0-1, 1}(\mathbb{R})$ otherwise.
3. If $0 < \alpha - \alpha_0 \leq 1$ then $f_\alpha \notin C^{1,0}(I)$ for any open interval I . The same is true for $\Re f_\alpha$ and $\Im f_\alpha$.

Theorem 2 (Local regularity at rationals). *Let x be any rational number and $\beta(x), \beta^*(x)$ and $\beta^{**}(x)$ the Hölder exponents of either f_α , $\Re f_\alpha$ or $\Im f_\alpha$. Then:*

1. If f is cuspidal at x then $\beta(x) = 2\alpha - r$. Otherwise $\beta(x) = \alpha - r$.

2. If f is a cusp form then

$$\beta^*(x) = [\alpha - r/2] + \min(1, 2\{\alpha - r/2\}).$$

If f is not a cusp form then

$$\beta^*(x) = \begin{cases} [\alpha - r] + \min(1, 2\{\alpha - r\} + r) & f \text{ cuspidal at } x, \alpha - r \notin \mathbb{Z} \\ \alpha - r & f \text{ not cuspidal at } x \text{ or } \alpha - r \in \mathbb{Z}. \end{cases}$$

3. In any case $\beta^{**}(x) = \alpha - \alpha_0$.

4. If $0 < \alpha - \alpha_0 \leq 1$ then f_α (resp. $\Re f_\alpha, \Im f_\alpha$) is not differentiable at any rational point which is not cuspidal for f . If x is cuspidal for f then f_α (resp. $\Re f_\alpha, \Im f_\alpha$) is differentiable at x if and only if $\alpha > (r+1)/2$, and in this case the derivative is given by

$$f'_\alpha(x) = \frac{(2\pi)^\alpha}{(im)^\alpha \Gamma(\alpha)} \int_{(x)} (z-x)^{\alpha-1} f'(z) dz.$$

The regularity at a particular irrational point depends on how well this point can be approximated by rationals which are not cuspidal for f . This is precisely measured by the following quantity:¹

$$\tau_x := \sup \left\{ \tau : \left| x - \frac{p}{q} \right| \ll \frac{1}{q^\tau} \text{ for infinitely many non-cuspidal rationals } \frac{p}{q} \right\}. \quad (6)$$

The inequality $\tau_x \geq 2$ is not only always satisfied for any irrational number x but, in fact, 2 is always contained in the set on the right hand side of (6). This follows from Hedlund's lemma (see [20]).

Theorem 3 (Local regularity at irrationals). *Let x be any irrational number and $\beta(x), \beta^*(x)$ and $\beta^{**}(x)$ the Hölder exponents of either $f_\alpha, \Re f_\alpha$ or $\Im f_\alpha$. Then:*

1. If f is a cusp form then $\beta(x) = \beta^*(x) = \beta^{**}(x) = \alpha - r/2$.

2. If f is not a cusp form,

$$\begin{aligned} \beta(x) &= \alpha - \left(1 - \frac{1}{\tau_x}\right) r \\ \beta^*(x) &= \begin{cases} [\alpha - r] + \min(1, \{\alpha - r\} + r/\tau_x) & \alpha - r \notin \mathbb{Z} \\ \alpha - r & \alpha - r \in \mathbb{Z} \end{cases} \\ \beta^{**}(x) &= \alpha - r. \end{aligned}$$

¹The symbol \ll could be replaced by \leq in this definition without affecting the value of τ_x , but this convention simplifies some arguments later on.

Remark. *Regarding the differentiability of these functions at irrational points we could not prove anything beside the obvious results: it is not differentiable whenever $\beta(x) < 1$, while it must be for $\beta(x) > 1$.*

As mentioned in the introduction an approximate functional equation plays a key role in the proof of some of these results. This equation has interest on its own and for this reason we include it here:

Theorem 4 (Approximate functional equation). *Let $\sigma = \sigma_{x_0}$ be any scaling matrix for the cusp $x_0 \in \mathbb{Q}$. Then:*

$$f_\alpha(x) = Ai^{-\alpha} f(x_0) \psi(x - x_0) + B|x - x_0|^{2\alpha} (x - x_0)^{-r} f_\alpha^\sigma(\sigma^{-1}x) + E(x)$$

where

$$\psi(x) = \begin{cases} x^{\alpha-r} & \alpha - r \notin \mathbb{Z} \\ x^{\alpha-r} \log x & \alpha - r \in \mathbb{Z} \end{cases}.$$

The constants A and B are real and nonzero, and moreover $B > 0$. The error term $E(x)$ is $C^{1,0}$ outside x_0 and satisfies $E \in \mathcal{C}^{2\alpha-r+1}(x_0)$.

This theorem is still true for any $\sigma \in \mathrm{SL}_2(\mathbb{R})$ as long as f^σ is a modular form for a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and the bottom-left entry of σ is negative (see §4). Note that it may be applied to $f^{\sigma_{\mathfrak{b}}}$ and $\sigma = \sigma_{\mathfrak{b}}^{-1} \sigma_{\mathfrak{a}}$ to link the behavior of $f_\alpha^{\sigma_{\mathfrak{b}}}$ near $\sigma(\infty)$ with the function $f_\alpha^{\sigma_{\mathfrak{a}}}$ for any two rational cusps \mathfrak{a} and \mathfrak{b} . Note also that if $\sigma \in \Gamma$ then the function f^σ coincides up to constant with f , and hence the result may be understood as an approximate version of (2), with an extra term appearing if $f(x_0) \neq 0$.

Theorem 4 is a generalization of lemma 3.8 of [6], and was already known in the literature in the case when f is a classical cusp form of integer weight $r > 2$ and $\alpha = r - 1$. In this context f_{r-1} is called the Eichler integral of f and the approximate equation is in fact exact, the error term corresponding to the period polynomial of f of the Eichler-Shimura theory (cf. [8]):

Corollary 5. *If f is a cusp form of weight $r > 2$ and $\alpha = r - 1$ then the error term $E(x)$ in theorem 4 is given by*

$$E(x) = \frac{(2\pi)^{r-1}}{(im)^{r-1} \Gamma(r-1)} \int_{x_0}^{x_0+i\infty} (z-x)^{r-2} f(z) dz.$$

If moreover r is an integer then E is a polynomial.

When f is not a cusp form the pointwise Hölder exponent of f_α at the irrational numbers ranges in a continuum between the values $\alpha - r$ and

$\alpha - r/2$ (cf. theorem 3). An interesting concept to study in this case is that of the spectrum of singularities, which consists of the function $d : [0, +\infty) \rightarrow [0, 1] \cup \{-\infty\}$ that assigns to each $\delta \geq 0$ the Hausdorff dimension of the set $\{x : \beta(x) = \delta\}$ if this set is nonempty and $-\infty$ otherwise (cf. [6, 17]). If the image of d is not discrete then it is said that f_α is a multifractal function, for the sets where a certain regularity is attained then correspond to a “continuous collection” of sets with fractional Hausdorff dimension.

Theorem 6 (Spectrum of singularities). *Let d be the spectrum of singularities of either f_α , $\Re f_\alpha$ or $\Im f_\alpha$. Then:*

1. *If f is a cusp form:*

$$d(\delta) = \begin{cases} 1 & \delta = \alpha - r/2 \\ 0 & \delta = 2\alpha - r \\ -\infty & \text{in other cases.} \end{cases}$$

2. *If f is not a cusp form:*

$$d(\delta) = \begin{cases} 2 + 2\frac{\delta - \alpha}{r} & \alpha - r \leq \delta \leq \alpha - r/2 \\ 0 & \delta = 2\alpha - r \\ -\infty & \text{in other cases.} \end{cases}$$

As a corollary the functions f_α , $\Re f_\alpha$ and $\Im f_\alpha$ are multifractal if and only if f is not a cusp form.

3 Preliminary lemmas

We include in this section some auxiliary results that will be used later on. The first ones describe some general aspects of the behavior of modular forms, and are consequence of (2) and (3). Their proofs are based on §3 of [5].

Lemma 7 (Expansion at the cusps). *Let p, q be coprime integers and $z = x + iy \in \mathbb{H}$. Let $m = m_{p/q}$ be the width of the cusp p/q . Suppose the quantity $|qz - p|^2/y$ remains uniformly bounded. Then:*

$$f(z) = \frac{f(p/q)}{m^{r/2}(qz - p)^r} + O\left(y^{-r/2}e^{-Ky|qz - p|^{-2}}\right)$$

for some constant $K > 0$ independent of p/q .

Proof. A scaling matrix $\sigma = \sigma_{p/q}$ for the cusp p/q has the following form:

$$\sigma^{-1} = \begin{pmatrix} * & * \\ q\sqrt{m} & -p\sqrt{m} \end{pmatrix}.$$

Once we have fixed $\delta > 0$, from (3) one deduces that for some $K' > 0$,

$$f(\sigma w) = (j_\sigma(w))^r f(p/q) + O(|j_\sigma(w)|^r e^{-K' \Im w}) \quad (\Im w \geq \delta).$$

We now perform the change of variables $\sigma w = z$ and use $j_\sigma(w) = (j_{\sigma^{-1}}(z))^{-1}$ and $\Im w = \Im z |j_{\sigma^{-1}}(z)|^{-2}$, from where the desired expansion follows at once. The constant K may be chosen independent of p/q because there are only finitely many equivalence classes of cusps. \square

The condition $|qz - p|^2/y \leq \delta$ that appeared in lemma 7 has the geometric meaning of imposing that z lies in the circle

$$\left\{ z : \left| z - \frac{p}{q} - i \frac{\delta}{2q^2} \right| \leq \frac{\delta}{2q^2} \right\} = \gamma(\{\Im z \geq \delta^{-1}\}), \quad \gamma = \begin{pmatrix} p & * \\ q & * \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

These circles are usually called generalized Ford circles (Speiser circles). In the particular case $\delta = 2$ we will denote them by $\mathcal{F}_{p/q}$ and use the following property: $\bigcup \mathcal{F}_{p/q} \supset \{0 < \Im z < 1/2\}$. This is clear from the fact that the fundamental domain for $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ is contained in $\{\Im z \geq 1/2\}$.²

We recall that we have defined α_0 as $r/2$ if f is a cusp form and r otherwise.

Lemma 8. *The modular form f satisfies*

$$f(z) \ll (\Im z)^{-\alpha_0} \quad (\Im z \rightarrow 0^+).$$

Conversely one has $f(x + iy) \gg y^{-r/2}$ whenever x is an irrational number and $f(x + iy) \gg y^{-r}$ whenever x is a non-cuspidal rational point, both bounds valid for infinitely many values of $y \rightarrow 0^+$ (not uniformly in x).

Proof. Let $z = x + iy$ with $0 < y < 1/2$. By the previous remarks z must be contained in some Ford circle $\mathcal{F}_{p/q}$, and hence lemma 7 shows that $f(z) \ll y^{-r}$. If f is cuspidal at p/q then by the same argument $f(z) \ll y^{-r/2}$; in particular this happens when f is a cusp form.

²The standard Ford circles (case $\delta = 1$) are intimately related to Farey sequences and diophantine approximation, as is beautifully explained in [10].

If x is a non-cuspidal rational point then by lemma 7 applied at $p/q = x$ we obtain $f(z) \gg y^{-r}$ as $y \rightarrow 0^+$.

(Lem. 3.4 of [5]) For the remaining case we consider the function $g(z) = y^{r/2}|f(z)|$. Since the multiplier μ_γ in (2) is unimodular it readily follows that g is Γ -invariant. Now if x is an irrational number then the line $\{\Re z = x\}$ cuts the boundary of infinitely many generalized Ford circles for any $\delta \geq 2$ at a sequence of points $x + iy_n$ with arbitrarily small y_n . For each of them we may find an element $\gamma \in \text{SL}_2(\mathbb{Z})$ sending $x + iy_n$ to a point w_n in the line $\{\Im w = \delta^{-1}\}$. We may further assume $-1/2 \leq \Re w_n \leq 1/2$ by composing γ with a translation if necessary. The inverse of γ can be decomposed as $\gamma^{-1} = \gamma' \gamma_i$, where $\gamma' \in \Gamma$ and γ_i pertains to a fixed finite right transversal. Therefore $g(x + iy_n) = g(\gamma_i(w_n))$, for some point w_n in the segment $I_\delta = \{\Im w = \delta^{-1}, -1/2 \leq \Re w \leq 1/2\}$. Since each of the functions $g_i(z) = g(\gamma_i(z))$ has a finite number of zeros in every compact subset of \mathbb{H} , we may choose δ so that none of the g_i vanish on I_δ , guaranteeing $g_i(w_n) \gg 1$. This proves $f(x + iy_n) \gg y_n^{-r/2}$. \square

Lemma 9. *Let $\tau \geq 2$ and x_0 an irrational number. The following holds:*

1. *If all the non-cuspidal rationals p/q satisfy*

$$\left| x_0 - \frac{p}{q} \right| \gg \frac{1}{q^\tau} \quad (7)$$

then $f(x + iy) \ll y^{-(1-\frac{1}{\tau})r} + y^{-r}|x - x_0|^{\frac{r}{\tau}}$ for $0 < y < 1/2$.

2. *If there are infinitely many non-cuspidal rationals p/q satisfying*

$$\left| x_0 - \frac{p}{q} \right| \ll \frac{1}{q^\tau} \quad (8)$$

then $f(x_0 + iy) \gg y^{-(1-\frac{1}{\tau})r}$ for infinitely many values of $y \rightarrow 0^+$.

Proof. 1) Let $z = x + iy$ with $0 < y < 1/2$. Then z must be contained in one of the circles $\mathcal{F}_{p/q}$. We will use again the expansion at the cusp given by lemma 7. If p/q is cuspidal for f then:

$$f(x + iy) \ll y^{-r/2} \leq y^{-(1-\frac{1}{\tau})r}.$$

If p/q is not cuspidal we have

$$f(x + iy) \ll q^{-r} \left(\left(x - \frac{p}{q} \right)^2 + y^2 \right)^{-r/2}.$$

By hypothesis p/q must satisfy (7) and therefore

$$q^{-r} \ll \left| x_0 - \frac{p}{q} \right|^{r/\tau} \ll \left| x - \frac{p}{q} \right|^{r/\tau} + |x - x_0|^{r/\tau}.$$

Hence:

$$f(x + iy) \ll \left| x - \frac{p}{q} \right|^{r/\tau} \left(\left(x - \frac{p}{q} \right)^2 + y^2 \right)^{-r/2} + y^{-r} |x - x_0|^{r/\tau}.$$

Arguing by cases depending on whether $y \leq |x - p/q|$ or not it is not hard to prove that the first term is $\ll y^{-(1-\frac{1}{\tau})r}$.

2) The case $\tau = 2$ has already been established in lemma 8, so we may assume $\tau > 2$. By hypothesis there must exist an equivalence class of non-cuspidal rationals modulo Γ for which infinitely many satisfy (8). For any of those rationals p/q we choose $z = x_0 + iy$ with $y = q^{-\tau}$ and note that

$$\frac{|qz - p|^2}{y} = q^{2+\tau} \left(\left| x_0 - \frac{p}{q} \right|^2 + y^2 \right) \ll q^{2-\tau}.$$

Applying lemma 7 again we obtain:

$$|f(x_0 + it)| = Cy^{-r/2} \left(\frac{y}{|qz - p|^2} \right)^{r/2} + O\left(y^{-r/2} e^{-Kq^{\tau-2}}\right) \gg q^{(\tau-1)r},$$

the constant C not depending on p/q . Since $q^{(\tau-1)r} = y^{-(1-\frac{1}{\tau})r}$ this finishes the proof. \square

Lemma 10. *The partial sums in the Fourier expansion (4) satisfy*

$$\sum_{n=0}^N a_n e^{2\pi i n x} \ll N^{\alpha_0} \log N.$$

Proof. (Lem. 3.2 of [5]) Using the Dirichlet kernel $D_N(z) = \sum_{|n| \leq N} e^{2\pi i n z}$ we may write

$$\left| \sum_{n=0}^N a_n e^{2\pi i n x} \right| \ll \int_0^1 |f(u + i/N)| |D_N(x - u - i/N)| du.$$

We apply lemma 8 to bound the first factor by N^{α_0} . Since $\|D_N\|_1 \ll \log N$ we obtain the estimate. \square

Lemma 11. *If f is a cusp form then the coefficients in the expansion (4) satisfy*

$$C_1 N^r \leq \sum_{n \leq N} |a_n|^2 \leq C_2 N^r$$

for some $C_1, C_2 > 0$.

Proof. (Lem. 3.2 of [5]) Let $g(z) = (\Im z)^{r/2} |f(z)|$. We claim that there are constants $C, C' > 0$ such that $|\{x : g(x + i/N) > C\} \cap [0, 1]| > C'$ for every integer $N \geq 0$. Because f is cuspidal the function g , being Γ -invariant and bounded in a fundamental domain for $\Gamma \backslash \mathbb{H}$ must be globally bounded. Using Parseval's identity,

$$N^r \ll N^r \int_0^1 |g(u + i/N)|^2 du = \sum_n |a_n|^2 e^{-4\pi n/N} \ll N^r.$$

The upper bound implies at once

$$\sum_{n \leq N} |a_n|^2 \ll N^r.$$

On the other hand,

$$\begin{aligned} \sum_{n \leq KN} |a_n|^2 &\geq \sum_n |a_n|^2 e^{-4\pi n/N} - \sum_{n > KN} |a_n|^2 e^{-4\pi n/N} \\ &\gg N^r - C'' e^{-2\pi K} N^r, \end{aligned}$$

where the sum after the minus sign has been estimated summing by parts and using the upper bound. We may now choose K big enough to finish the proof.

We still have to justify the previous claim. Let $C_1, C_2 > 0$ be constants to be determined later and consider the intervals $|x - p/q| \leq C_2/(qN^{1/2})$ with $C_1 N^{1/2} < q < C_2 N^{1/2}$. For $2C_2^3 < C_1$ these are disjoint and cover a positive portion of the interval $[0, 1]$. Suppose that $z = x + i/N$ with x lying in one of those intervals and let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ satisfying $\gamma(p/q) = \infty$. We may decompose $\gamma^{-1} = \gamma' \gamma_i$, where $\gamma' \in \Gamma$ and γ_i lies in a fixed finite right-transversal for Γ . Hence $g(z) = g_i(\gamma z)$ where $g_i(z) = g(\gamma_i z)$. It can be readily checked that $1/(2C_2^2) \leq \Im(\gamma z) \leq 1/C_1^2$, hence it suffices to show that we may choose C_1 and C_2 to ensure that every g_i is bounded below in that strip. But this follows from the fact that $g_i(z)/(\Im z)^{r/2} = |f^{\gamma_i}(z)|$ has a Fourier expansion (3). \square

The following lemma will be our main tool to study f_α . We do not assume $m = m_\infty = 1$ and $\kappa = \kappa_\infty = 0$ because we will need the general statement in §4.

Lemma 12. For $\alpha > \alpha_0$ the series (5) converges uniformly to a continuous function f_α , which admits the following integral representation

$$f_\alpha(x) = \frac{(2\pi)^\alpha}{(im)^\alpha \Gamma(\alpha)} \int_{(x)} (z-x)^{\alpha-1} (f(z) - f(\infty)) dz,$$

where (x) denotes the vertical ray connecting x with $i\infty$.

Proof. Summing by parts (5) and using the estimates for partial sums given in lemma 10 it is plain that the series converges uniformly and hence to a continuous function.

To prove the integral representation we start with

$$f_\alpha(x+iy) = \frac{(2\pi)^\alpha}{m^\alpha \Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (f(x+iy+it) - f(\infty)) dt,$$

identity that can be obtained from (4) integrating the series term by term because of the uniform convergence in the region $\Im z \geq y$. Now it suffices to take the limit $y \rightarrow 0^+$ on both sides. The left hand side corresponds to the Abel summation of a converging Fourier series, while in the right hand side the dominated convergence theorem applies with the bounds obtained in lemma 8. \square

Our last lemma is a very particular version of the differentiation under the integral sign theorem.

Lemma 13. Let $\gamma \in \text{SL}_2(\mathbb{R})$ and let I be a bounded open interval whose closure does not contain the pole of γ . Let $g(z, x)$ be a function continuously differentiable with respect to x in I and analytic for $z \in \mathbb{H}$. Assume moreover that both g and g_x have exponential decay when $\Im z \rightarrow +\infty$ in vertical strips, and that for some $\beta > 0$, $\eta > 0$ they satisfy the following estimates when $z \rightarrow \gamma(x)$ uniformly in $x \in I$:

$$\begin{aligned} g(z, x) &= O((z - \gamma x)^{\beta+\eta-1} (\Im z)^{-\eta}) \\ g_x(z, x) &= O((z - \gamma x)^{\beta+\eta-2} (\Im z)^{-\eta}) \end{aligned}$$

Then the function

$$F(x) = \int_{(\gamma x)} g(z, x) dz \quad (x \in I)$$

is in $\Lambda_\beta(I)$ for $0 < \beta < 1$, in $\Lambda_1^{\log}(I)$ for $\beta = 1$ and in $C^{1,0}(I)$ for $\beta > 1$. In this last case,

$$F'(x) = \int_{(\gamma x)} g_x(z, x) dz \quad (x \in I).$$

Proof. Assume $x \in I$ and $h \neq 0$ satisfying $x + h \in I$. Using Cauchy's theorem together with the estimates for g we can write for $0 < u < v$:

$$F(x+h) - F(x) = \int_{\gamma(x)+iu}^{\gamma(x)+iv} (g(z, x+h) - g(z, x)) dz + O\left(e^{-Kv} + u^\beta + hu^{\beta-1} + \frac{h^{\beta+\eta}}{u^\eta}\right).$$

It is clear now that F must be continuous, as for each ε we may choose u and v so that for h small enough $|F(x+h) - F(x)| \leq \varepsilon$.

For the rest of the proof we choose $u = h$ and $v = +\infty$, so that the error term is of the form $O(h^\beta)$. By the mean value theorem:

$$|F(x+h) - F(x)| \ll h \int_{\gamma(x)+ih}^{\gamma(x)+i\infty} |g_x(z, x_z)| |dz| + O(h^\beta).$$

Using the estimates for g_x this last integral is of order $O(h^{\beta-1})$ for $0 < \beta < 1$ and of order $O(\log h)$ for $\beta = 1$.

Suppose now that $\beta > 1$. The estimates for g_x justify the use of the dominated convergence theorem, proving the existence and the formula for F' . Finally, the argument used to prove that F is continuous can be applied directly to F' substituting β by $\beta - 1$ to conclude that F' is also continuous. \square

4 Approximate functional equation

Let $\sigma \in \mathrm{SL}_2(\mathbb{R})$ such that f^σ is a modular form for a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and assume its bottom-left entry is negative. Fix $x_0 = \sigma(\infty) \in \mathbb{Q}$ and let x be an arbitrary real number distinct from x_0 . Moreover set $C_0 = (2\pi)^\alpha / (i^\alpha \Gamma(\alpha))$.

Our starting point will be the integral representation given by lemma 12:

$$f_\alpha(x) = C_0 \int_{(x)} (z-x)^{\alpha-1} (f(z) - f(\infty)) dz. \quad (9)$$

For the sake of simplicity we will hide some extra terms that appear along the way inside the symbol (\cdots) ; we will deal with them afterwards. The reader can check that they all appear in (10-13).

Splitting the integral and performing the change of variables $z = \sigma w$ in (9) we have:

$$\begin{aligned}
f_\alpha(x) &= C_0 \int_x^{x+2i} (z-x)^{\alpha-1} f(z) dz + (\dots) \\
&= C_0 \int_S (\sigma w - x)^{\alpha-1} (j_\sigma(w))^{r-2} (f^\sigma(w) - f(x_0)) dw + (\dots).
\end{aligned}$$

where S corresponds to a subarc of the halfcircle with endpoints $\sigma^{-1}(x)$ and $\sigma^{-1}(\infty)$. If σ is not a scaling matrix then by $f(x_0)$ we mean the constant $f^\sigma(\infty)$, which might differ from the one introduced in §2. If σ is a scaling matrix then both definitions agree.

The integrand in the last equation has exponential decay when $\Im w \rightarrow +\infty$. Applying Cauchy's theorem to replace S with two vertical rays starting at the endpoints of S and projecting to $i\infty$:

$$f_\alpha(x) = C_0 \int_{(\sigma^{-1}x)} (\sigma w - x)^{\alpha-1} (j_\sigma(w))^{r-2} (f^\sigma(w) - f(x_0)) dw + (\dots).$$

Substituting the relation $(\sigma w - x)j_\sigma(w) = (w - \sigma^{-1}x)j_{\sigma^{-1}}(x)$ [16, (2.4)]:

$$f_\alpha(x) = C_0 C_1 (j_{\sigma^{-1}}(x))^{\alpha-1} \int_{(\sigma^{-1}x)} (w - \sigma^{-1}x)^{\alpha-1} (j_\sigma(w))^{r-\alpha-1} (f^\sigma(w) - f(x_0)) dw + (\dots).$$

The constant C_1 has the value $e^{-2\pi i \alpha}$ if $x < x_0$ and 1 otherwise. This integral is of the form given in lemma 12 except for the extra factor $\phi(w) = (j_\sigma(w))^{r-\alpha-1}$. We will denote by $\phi(u^+)$ the limit of $\phi(w)$ when $w \rightarrow u$ from the upper half-plane. Adding and subtracting $\phi(\sigma^{-1}x^+) = (j_{\sigma^{-1}}(x))^{\alpha-r+1}$ in the integrand, and using that $j_{\sigma^{-1}}(x) = B_1(x - x_0)$ for some $B_1 > 0$, we arrive to

$$f_\alpha(x) = B|x - x_0|^{2\alpha}(x - x_0)^{-r} f_\alpha^\sigma(\sigma^{-1}x) + (\dots).$$

The terms we have ignored are the following ones:

$$(\dots) = -C_0 \frac{(2i)^\alpha}{\alpha} f(\infty) + C_0 \int_{x+2i}^{x+i\infty} (z-x)^{\alpha-1} (f(z) - f(\infty)) dz \quad (10)$$

$$+ C_0 f(x_0) \int_x^{x+2i} (z-x)^{\alpha-1} (j_{\sigma^{-1}}(z))^{-r} dz \quad (11)$$

$$+ C_0 \left(\int_{x_0}^{x_0+2i} + \int_{x_0+2i}^{x+2i} \right) (z-x)^{\alpha-1} \left(f(z) - \frac{f(x_0)}{(j_{\sigma^{-1}}(z))^r} \right) dz \quad (12)$$

$$+ C(x - x_0)^{\alpha-1} \int_{(\sigma^{-1}x)} (w - \sigma^{-1}x)^{\alpha-1} (\phi(w) - \phi(\sigma^{-1}x^+)) (f^\sigma(w) - f(x_0)) dw. \quad (13)$$

The terms (10) and (12) make sense for any $x \in \mathbb{R}$ and are infinitely many times differentiable with respect to this variable. The other ones are dealt with in the following lemmas:

Lemma 14. *The term (11) has the following expansion:*

$$(11) = Ai^{-\alpha}f(x_0)\psi(x - x_0) + E(x)$$

where

$$\psi(x) = \begin{cases} x^{\alpha-r} & \alpha - r \notin \mathbb{Z} \\ x^{\alpha-r} \log x & \alpha - r \in \mathbb{Z} \end{cases}.$$

The constant $A \neq 0$ is real and the error term $E(x)$ is infinitely many times differentiable.

Lemma 15. *The term (13) is $C^{1,0}$ outside x_0 and of type $O(|x - x_0|^{2\alpha-r+1})$ when $x \rightarrow x_0$.*

These two lemmas together with the previous manipulations constitute the proof of theorem 4.

Proof of lemma 14. We may assume that f is not cuspidal at x_0 , since otherwise (11) is equal to zero. Note that in this case by hypothesis $\alpha > r$. Renaming $x - x_0$ to x if necessary we may further assume $x_0 = 0$. Hence up to a nonzero constant of the form $Ai^{-\alpha}f(x_0)$ we have to expand asymptotically the function

$$g(x) = \int_0^{2i} \frac{z^{\alpha-1}}{(x+z)^r} dz. \quad (14)$$

We will suppose for the moment that $x > 0$ and $\alpha - r \notin \mathbb{Z}$. We have

$$g(x) = x^{-r} \int_0^{2xi} \frac{z^{\alpha-1}}{\left(1 + \frac{z}{x}\right)^r} dz + \int_{2xi}^{2i} \frac{z^{\alpha-r-1}}{\left(1 + \frac{x}{z}\right)^r} dz.$$

In the first integral we perform a linear change of variables, while in the second one we substitute the Laurent expansion

$$\left(1 + \frac{x}{z}\right)^{-r} = \sum_{k \geq 0} \binom{-r}{k} x^k z^{-k}$$

which is uniformly convergent in the region $|z| \geq 2x$. Integrating term by term the expression now results

$$\begin{aligned} g(x) &= x^{\alpha-r} \int_0^{2i} \frac{z^{\alpha-1}}{(1+z)^r} dz + \sum_{k \geq 0} \binom{-r}{k} \frac{x^k}{\alpha-r-k} z^{\alpha-r-k} \Big|_{2xi}^{2i} \\ &= x^{\alpha-r} \left(\int_0^{2i} \frac{z^{\alpha-1}}{(1+z)^r} dz - \sum_{k \geq 0} \binom{-r}{k} \frac{(2i)^{\alpha-r-k}}{\alpha-r-k} \right) + h(x). \end{aligned} \quad (15)$$

where $h(x)$ is a function given by a power series which converges in a neighborhood of 0. Notice that the expression within brackets is a constant A' satisfying

$$A' = \int_0^T \frac{z^{\alpha-1}}{(1+z)^r} dz - \sum_{k \geq 0} \binom{-r}{k} \frac{T^{\alpha-r-k}}{\alpha-r-k}$$

for any complex T with $|T| > 1$ and $\arg T \neq \pi$: the right hand side is indeed constant as can be easily checked by differentiating with respect to T . Hence

$$\begin{aligned} A' &= \lim_{T \rightarrow +\infty} \left(\int_0^T \frac{t^{\alpha-1}}{(1+t)^r} dt - \sum_{0 \leq k < \alpha-r} \binom{-r}{k} \frac{T^{\alpha-r-k}}{\alpha-r-k} \right) \\ &= \int_0^\infty t^{\alpha-1} \left(\frac{1}{(1+t)^r} - \sum_{0 \leq k < \alpha-r} \binom{-r}{k} \frac{1}{t^{r+k}} \right) dt. \end{aligned}$$

The sum corresponds to the Taylor expansion of order $[\alpha-r]$ of the function $(1-\xi)^{-r}$ multiplied by ξ^r and evaluated at $\xi = 1/t$. Since all the derivatives of this function have constant sign for $\xi > 0$ we deduce $A' \neq 0$. Although the exact value of A' is unimportant, using the integral formula for the error term in the Taylor expansion one can easily obtain a closed formula in terms of beta functions.

Suppose now that $\alpha-r$ is an integer. The same argument can be carried on, but when integrating the Laurent series term by term the term corresponding to $k = \alpha-r$ is now transformed into a logarithm. This term results

$$\binom{-r}{\alpha-r} x^{\alpha-r} \log z \Big|_{2xi}^{2i} = \binom{-r}{\alpha-r} x^{\alpha-r} (-\log(x/i) + \log 2 - \log T) \quad (T = 2i).$$

The first summand corresponds to the main term, while the other two should be merged into A' . This is relevant, as we will need $A' \in \mathbb{R}$ in order to handle

the case $x < 0$. We may replace (15) with:

$$g(x) = -\binom{-r}{\alpha-r} x^{\alpha-r} \log(x/i) + A' x^{\alpha-r} + h(x). \quad (16)$$

Finally if $x < 0$, we go back to (14) and notice that

$$g(x) = (-1)^{\alpha-r} \overline{g(-x)},$$

and the very same equation is also satisfied by the main and error terms in equations (15-16). Therefore we may apply the results we have obtained for $x > 0$. \square

Proof of lemma 15. Because of the extra cancelation as $w \rightarrow \sigma^{-1}x$ provided by the second factor inside the integral in (13) and the exponential decay given by the third factor when $\Im z \rightarrow +\infty$, lemma 13 can be applied with $\eta = \alpha_0$ and $\beta + \eta = \alpha + 1$. This shows that (13) is $C^{1,0}$ in the complement of $\{x_0\}$.

For the second estimate, it suffices to show that

$$\int_{(\sigma^{-1}x)} (w - \sigma^{-1}x)^{\alpha-1} (\phi(w) - \phi(\sigma^{-1}x^+)) (f^\sigma(w) - f(x_0)) dw \ll |x - x_0|^{\alpha-r+2} \quad (17)$$

when $x \rightarrow x_0$. Notice that for $w = \sigma^{-1}x + it$ we have

$$\phi(w) = (j_\sigma(w))^{r-\alpha-1} = \left(\frac{1}{(-c)(x-x_0)} + ict \right)^{r-\alpha-1}$$

where c is the bottom-left entry of σ . Therefore applying the mean value theorem we obtain for $|x - x_0| \leq 1$:

$$|\phi(w) - \phi(\sigma^{-1}x^+)| \ll \begin{cases} t|x-x_0|^{\alpha-r+2} & t \leq |x-x_0|^{-1} \\ t^{r-\alpha-1} & t \geq |x-x_0|^{-1} \end{cases}.$$

We divide now the integration domain in three intervals and use these estimates, together with the trivial ones for f^σ , concluding that the left hand side of (17) is

$$\begin{aligned} &\ll |x - x_0|^{\alpha-r+2} \left(\int_0^1 t^\alpha (1 + t^{-\alpha_0}) dt + \int_1^{|x-x_0|^{-1}} t^\alpha e^{-Kt} dt \right) \\ &\quad + \int_{|x-x_0|^{-1}}^\infty t^{r-2} e^{-Kt} dt. \end{aligned}$$

This proves (17), since the first two integrals are convergent and the last one has exponential decay when $x \rightarrow x_0$. \square

Proof of corollary 5. If f is a cusp form then (11) and the first summand of (10) vanish. Moreover since $\alpha = r - 1$ the function ϕ in (13) is constant, and hence this term also vanishes. The remaining terms are:

$$\begin{aligned} (\cdots) &= C_0 \left(\int_{x_0}^{x_0+2i} + \int_{x_0+2i}^{x+2i} + \int_{x+2i}^{x+i\infty} \right) (z-x)^{\alpha-1} f(z) dz \\ &= \frac{(2\pi)^\alpha}{i^\alpha \Gamma(\alpha)} \int_{(x_0)} (z-x)^{\alpha-1} f(z) dz. \end{aligned} \quad \square$$

5 Wavelet transform

Let $\alpha > 0$ and suppose that ψ is a function satisfying:

1. $\psi^{(k)}(x) \ll (1+|x|)^{-\alpha-1}$ for all $k \geq 0$.
2. $\int_{\mathbb{R}} x^k \psi(x) dx = 0$ for $0 \leq k < \alpha$.
3. Either

$$\int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} = \int_0^\infty |\hat{\psi}(-\xi)|^2 \frac{d\xi}{\xi} = 1$$

or

$$\hat{\psi}(\xi) = 0 \text{ if } \xi < 0 \quad \text{and} \quad \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} = 1.$$

Then we say that ψ is an analytic wavelet, and we define the wavelet transform of a bounded function f with respect to ψ as

$$W(a, b) := \frac{1}{a} \int_{\mathbb{R}} f(t) \bar{\psi} \left(\frac{t-b}{a} \right) dt \quad (b \in \mathbb{R}, a > 0).$$

If we also ask f to be periodic, with vanishing integral on each period, and satisfying $\hat{f}(\xi) = 0$ for $\xi < 0$ in the distributional sense in case the same is satisfied by ψ , then we have the following inversion formula:

$$f(x) = \iint_{\mathbb{R} \times \mathbb{R}^+} W(a, b) \psi \left(\frac{x-b}{a} \right) \frac{db da}{a^2}. \quad (18)$$

The proof of this fact can be found in [14]. The outer integral in (18) in principle has to be understood as an improper Riemann integral, but in our applications it will be absolutely convergent.

The interesting fact about this transform is that the regularity of f in a point x_0 can be read off from the behaviour of its wavelet transform W in a neighborhood of the point $(0^+, x_0)$, as it is shown in the following two theorems, which are slight modifications of the ones appearing in [18].

Theorem 16. *Let $0 < \beta < \alpha$. If $f \in \mathcal{C}^\beta(x_0)$ then*

$$W(a, b) \ll a^\beta + |b - x_0|^\beta$$

when $(a, b) \rightarrow (0^+, x_0)$.

Theorem 17. *Let $0 < \beta' < \beta < \alpha$. If*

$$W(a, b) \ll a^\beta + a^{\beta-\beta'} |b - x_0|^{\beta'}$$

when $(a, b) \rightarrow (0^+, x_0)$ then $f \in \mathcal{C}^\beta(x_0)$ if β is not an integer and $f \in \mathcal{C}_{\log}^\beta(x_0)$ otherwise.

The bounds involving $W(a, b)$ in these two theorems may also be written in the forms $a^\beta \left(1 + \frac{|b-x_0|}{a}\right)^\beta$ and $a^\beta \left(1 + \frac{|b-x_0|}{a}\right)^{\beta'}$, respectively, from where it is clear that the second one constitutes a strengthening of the first.

Remark. *Theorem 17 as stated in [18] is not entirely true: the author omits that in the special case $\beta \in \mathbb{Z}$ the function f can indeed have a “logarithmic chirp”, i.e., $f(x) \approx (x - x_0)^\beta \log |x - x_0|$ when $x \rightarrow x_0$ (as it is shown by theorem 4; cf. §6). In the mentioned paper the proof is left to the reader for $\beta \geq 1$, and the range of applicability is somewhat more limited, as Jaffard employs a broader definition of analytic wavelet for which this theorem is only true if one adds the extra hypothesis $[\beta] \leq \alpha - 1$.*

Proof of theorem 16. We can assume without loss of generality $x_0 = 0$. By hypothesis there is a polynomial P of degree strictly smaller than α such that

$$|f(x) - P(x)| \ll |x|^\beta,$$

estimate which we may assume to hold globally. Hence, by the property 2 of analytic wavelets,

$$\begin{aligned} W(a, b) &\ll \frac{1}{a} \int_{\mathbb{R}} |f(t) - P(t)| \left| \psi\left(\frac{t-b}{a}\right) \right| dt \\ &\ll \frac{1}{a} \int_{\mathbb{R}} \frac{|t|^\beta}{\left(\left|\frac{t-b}{a}\right| + 1\right)^{\alpha+1}} dt \\ &\ll a^\beta \int_{\mathbb{R}} \frac{|t|^\beta}{(|t| + 1)^{\alpha+1}} dt + |b|^\beta \int_{\mathbb{R}} \frac{dt}{(|t| + 1)^{\alpha+1}} \\ &\ll a^\beta + |b|^\beta. \end{aligned} \quad \square$$

In order to prove theorem 17 we shall use the inversion formula (18), which for convenience will be written in the following way:

$$f(x) = \int_{\mathbb{R}^+} \omega(a, x) \frac{da}{a} \quad (19)$$

where

$$\omega(a, x) = \frac{1}{a} \int_{\mathbb{R}} W(a, b) \psi\left(\frac{x-b}{a}\right) db. \quad (20)$$

We prove first some estimates for ω . In particular they show that (19) is an absolutely convergent integral.

Lemma 18. *Under the hypothesis of theorem 17 the function $x \mapsto \omega(a, x)$ is infinitely many times differentiable and satisfies for all $k \geq 0$ and for some $\delta > 0$:*

$$\frac{\partial^k \omega}{\partial x^k}(a, x) \ll a^{-k-1}, \quad (21)$$

$$\frac{\partial^k \omega}{\partial x^k}(a, x) \ll a^{\beta-k} + a^{\beta-\beta'-k} |x - x_0|^{\beta'} \quad (a \leq 1, |x - x_0| \leq \delta) \quad (22)$$

Proof. It is clear that $W(a, b)$ is uniformly bounded and ψ and all its derivatives have decay (property 1 of analytic wavelets). Therefore we may differentiate (20) under the integral sign obtaining

$$\frac{\partial^k \omega}{\partial x^k}(a, x) = \frac{1}{a^{k+1}} \int_{\mathbb{R}} W(a, b) \psi^{(k)}\left(\frac{x-b}{a}\right) db. \quad (23)$$

Integrating by parts in the definition of $W(a, b)$ and using that the integral over each period of f vanishes it is readily seen that $W(a, b) \ll a^{-1}$. Plugging this into (23) one obtains (21).

To prove (22) we first assume without loss of generality that $x_0 = 0$, and that the bounds in the statement of theorem 17 hold uniformly in the neighborhood $a \leq 1$ and $|b| \leq 2\delta$. We have for $a \leq 1$ and $|x| \leq \delta$:

$$\begin{aligned} \frac{\partial^k \omega}{\partial x^k}(a, x) &\ll \frac{1}{a^{k+1}} \int_{|b| \leq 2\delta} \frac{a^\beta + a^{\beta-\beta'} |b|^{\beta'}}{\left(\left|\frac{x-b}{a}\right| + 1\right)^{\alpha+1}} db + \frac{1}{a^{k+1}} \int_{|b| > 2\delta} \frac{db}{\left(\left|\frac{x-b}{a}\right| + 1\right)^{\alpha+1}} \\ &\ll a^{\beta-k} + a^{\beta-\beta'-k} \int_{\mathbb{R}} \frac{|x-at|^{\beta'}}{(|t|+1)^{\alpha+1}} dt + \frac{1}{a^k} \int_{t > \delta/a} \frac{dt}{(t+1)^{\alpha+1}} \\ &\ll a^{\beta-k} + a^{\beta-\beta'-k} |x|^{\beta'}. \end{aligned} \quad \square$$

Proof of theorem 17. Again we can assume $x_0 = 0$. Let $N = [\beta]$ if β is not an integer and $N = \beta - 1$ otherwise. We perform a Taylor expansion of order N on ω :

$$\omega(a, x) = \sum_{k=0}^N \frac{\partial^k \omega}{\partial x^k}(a, 0) \frac{x^k}{k!} + E(a, x).$$

Using the bounds of lemma 18 we can plug this into (19) to obtain

$$f(x) = P(x) + \int_{\mathbb{R}^+} E(a, x) \frac{da}{a}$$

for certain polynomial P of degree at most $[\beta]$. It suffices to prove that the integral term has the right behavior when $x \rightarrow 0$.

We split the integral. In the range $a \leq |x|$ we use (22) with either $x = 0$ or $k = 0$ to obtain

$$\left| \int_{a \leq |x|} E(a, x) \frac{da}{a} \right| \leq \int_{a \leq |x|} |\omega(a, x)| \frac{da}{a} + \sum_{k=0}^N \frac{|x|^k}{k!} \int_{a \leq |x|} \left| \frac{\partial^k \omega}{\partial x^k}(a, 0) \right| \frac{da}{a} \ll |x|^\beta.$$

In the complementary range, assuming that β is not an integer, we use the formula for the Taylor error term together with (22):

$$\left| \int_{a \geq |x|} E(a, x) \frac{da}{a} \right| \leq \frac{|x|^{N+1}}{(N+1)!} \int_{a \geq |x|} \left| \frac{\partial^{N+1} \omega}{\partial x^{N+1}}(a, \xi_{a,x}) \right| \frac{da}{a} \ll |x|^\beta.$$

When β is an integer the same argument works using (22) in the range $|x| \leq a \leq 1$ and (21) in the range $a \geq 1$. The right hand side has to be replaced by $|x|^\beta \log |x|$. \square

We will apply these theorems with $\psi(x) = (x+i)^{-\alpha-1}$ to f_α , where f is a modular form. The reader can easily verify that ψ satisfies properties 1 and 2 of analytic wavelets. In order to check property 3 we compute $\hat{\psi}$. The integral

$$\hat{\psi}(\xi) = \int_{\mathbb{R}} \frac{e^{-2\pi i \xi x}}{(x+i)^{\alpha+1}} dx$$

vanishes for $\xi \leq 0$ by Cauchy's theorem. For $\xi > 0$ we perform a change of variables obtaining

$$\hat{\psi}(\xi) = \xi^\alpha e^{-2\pi \xi} \int_{\mathbb{R}+\xi i} \frac{e^{-2\pi i z}}{z^{\alpha+1}} dz$$

and by Cauchy's theorem the integral on the right hand side is a constant with respect to ξ . The exact value of the constant is not important, since

ψ needs not to be normalized for theorems 16 and 17 to hold, although it can be explicitly computed by means of Hankel's contour integral for the reciprocal of the gamma function (*cf.* [25]).

It is also clear that f_α is a periodic function, with vanishing integral in each period, and whose Fourier transform in the distributional sense is supported only in the positive frequencies. To compute its wavelet transform with respect to ψ we just need to compute the one for $g(x) = e^{2\pi i \lambda x}$. This can be done using some basic properties of the Fourier transform:

$$W_g(a, b) = e^{2\pi i \lambda b} \hat{\psi}(\lambda a) = \begin{cases} C a^\alpha \lambda^\alpha e^{2\pi i \lambda (b + ai)} & \lambda > 0 \\ 0 & \lambda \leq 0. \end{cases} \quad (24)$$

Hence

$$W_{f_\alpha}(a, b) = C a^\alpha (f(b + ai) - f(\infty)). \quad (25)$$

Corollary 19. *If for some $0 < \beta < \alpha$ one has $f_\alpha \in \mathcal{C}^\beta(x_0)$ then*

$$f(b + ai) \ll a^{\beta-\alpha} + a^{-\alpha} |b - x_0|^\beta$$

when $(a, b) \rightarrow (0^+, x_0)$. Reciprocally, if for some $0 < \beta' < \beta < \alpha$ one has

$$f(b + ai) \ll a^{\beta-\alpha} + a^{\beta-\beta'-\alpha} |b - x_0|^{\beta'}$$

when $(a, b) \rightarrow (0^+, x_0)$, then $f_\alpha \in \mathcal{C}^\beta(x_0)$ if β is not an integer and $f_\alpha \in \mathcal{C}_{\log}^\beta(x_0)$ otherwise. Moreover both statements remain true if one replaces f_α by its real or imaginary parts.

Proof. The part of the theorem concerning f_α follows at once from theorems 16 and 17 and (25). Also note that if $f_\alpha \in \mathcal{C}^\beta(x_0)$ or $f_\alpha \in \mathcal{C}_{\log}^\beta(x_0)$ then the same must hold for the real and the imaginary parts of f_α .

On the other hand, $\Re f_\alpha$ and $\Im f_\alpha$ are bounded functions, and hence their wavelet transforms are well defined. By rewriting the sine and cosine functions involved in their Fourier series by sums of exponentials and applying (24) one obtains

$$W_{f_\alpha}(a, b) = 2W_{\Re f_\alpha}(a, b) = 2iW_{\Im f_\alpha}(a, b).$$

And since the inversion formula (18) is not used in the proof of theorem 16, we may also apply this theorem to $\Re f_\alpha$ and $\Im f_\alpha$. \square

6 Regularity theorems

The aim of this section is proving theorems 1, 2 and 3.

Lemma 20. *If f_α is in $\mathcal{C}^{k,0}(x)$ for $k < \alpha - \alpha_0$, but cannot be continuously differentiated $k + 1$ times in any open interval containing x , then*

$$\beta^*(x) = k + \min\{1, \beta_{\alpha-k}(x)\},$$

where $\beta_{\alpha-k}$ denotes the pointwise Hölder exponent of $f_{\alpha-k}$. This formula extends to $\Re f_\alpha$ and $\Im f_\alpha$ if both these functions satisfy the hypothesis and their pointwise Hölder exponents coincide.

Proof. This follows at once from $f_\alpha^{(k)} = (2\pi i)^k f_{\alpha-k}$ and the definition of β^* . \square

Remark. *In order to prove theorems 1 and 2 we anticipate two very simple results which will come in handy. Applying corollary 19 with the bounds from lemma 8 we obtain $\beta(x) = \alpha - r/2$ for f cuspidal and x irrational and $\beta(x) = \alpha - r$ for f not cuspidal and x any non-cuspidal rational.*

Proof of theorem 1. 1) (Proposition 3.1 of [5]) If the series defining f_α converge at a certain point for $\alpha < \alpha_0$ then summing by parts the series defining f_{α_0} must also converge at that point, and therefore we may reduce to this case.

Suppose first that f is cuspidal, we will prove that $f_{r/2}$ diverges at any irrational point x . Considering the kernels of summability $\varphi_1(u) = e^{-2\pi u}(u^{r/2} + 1)$ and $\varphi_2(u) = e^{-2\pi u}$, we have (see Th. III.1.2 of [26]):

$$\lim_{y \rightarrow 0^+} y^{r/2} f(x + iy) = \lim_{t \rightarrow 0^+} \left(\sum_{n>0} A_n \varphi_1(ny) - \sum_{n>0} A_n \varphi_2(ny) \right) = 0$$

with $A_n = \frac{a_n}{n^{r/2}} e^{2\pi i n x}$, as long as $f_{r/2}$ converges at x ; but this contradicts lemma 8.

Suppose now that f is not cuspidal. We prove that f_r is not Abel summable at any non-cuspidal rational point x . If this were not the case then by lemma 12 we would have for some $\ell \in \mathbb{C}$,

$$\ell = \lim_{y \rightarrow 0^+} f_r(x + iy) = \lim_{y \rightarrow 0^+} \frac{(2\pi)^\alpha}{\Gamma(\alpha)} \int_y^\infty (t - y)^{r-1} (f(x + it) - f(\infty)) dt.$$

But since by the expansion at the cusp the term $f(x + it)$ behaves like Ct^{-r} for small t , the right hand side diverges.

2) The result follows from applying lemma 13 to the integral representation given by lemma 12 repeatedly.

3) Suppose first that f is not cuspidal. If $\alpha - r < 1$ then neither f_α nor its real or imaginary parts are differentiable at any non-cuspidal rational, since they are at most $(\alpha - r)$ -Hölder at these points. Only the limit case $\alpha = r + 1$ remains. But now we may appeal to theorem 4, since $2\alpha - r = r + 2 > 1$ implies that both the second term and the error term are differentiable at the rational x_0 , but the first term is not if x_0 is non-cuspidal. A more detailed analysis shows that neither the real nor the imaginary parts of the function $Cx \log x$ are differentiable at 0 for any complex constant C .

(Lem. 3.8 of [6]) Suppose now that f is cuspidal. If f_α is in $\mathcal{C}^{1,0}(I)$ then by theorem 4 it is also in $\mathcal{C}^{1,0}(\gamma(I))$ for any $\gamma \in \Gamma$. It follows that f'_α must exist and be continuous everywhere, and by Bessel's inequality

$$\|f'_\alpha\|_2^2 \gg \sum_{n>0} \frac{|a_n|^2}{n^{2\alpha-2}}.$$

But the right hand side diverges for $\alpha - r/2 \leq 1$ as can be checked by summing by parts and using the estimates of lemma 11.

Finally assume that either $\Re f_\alpha$ or $\Im f_\alpha$ is in $\mathcal{C}^{1,0}(I)$. Since the periodic Hilbert transform preserves the Sobolev space H^1 (cf. §3 of [15]) and sends a Fourier series to its conjugate series (and therefore $\Re f_\alpha$ to $\Im f_\alpha$ and $\Im f_\alpha$ to $-\Re f_\alpha$, cf. §II.5 of [26]), the function f_α must, at least, have a weak derivative in $L^2(I')$ for some smaller interval I' . This is enough to carry on the previous argument. \square

Proof of theorem 2. Let x_0 be a rational number.

1) If f is not cuspidal at x_0 then we already know $\beta(x_0) = \alpha - r$. Hence may assume that f is cuspidal at x_0 . Choose a scaling matrix σ satisfying $\sigma(\infty) = x_0$ and apply theorem 4. We deduce that $f_\alpha \in \mathcal{C}^{2\alpha-r}(x_0)$ and that $f_\alpha \notin \mathcal{C}^{2\alpha-r+\varepsilon}(x_0)$ for any $\varepsilon > 0$, since the term $\sigma^{-1}x$ diverges to ∞ when $x \rightarrow x_0$ and f_α^σ is a nonzero periodic function. Hence $\beta(x_0) = 2\alpha - r$. The same must be true for $\Re f_\alpha$ and $\Im f_\alpha$ as long as the image $f_\alpha^\sigma(\mathbb{R})$ is not contained in any one-dimensional subspace of \mathbb{C} . This is indeed the case as f_α^σ corresponds to a Fourier series with only positive frequencies.

2) The exponent β^* is determined by applying lemma 20 with $k = [\alpha - \alpha_0]$ if $\alpha - \alpha_0 \notin \mathbb{Z}$ and $k = \alpha - \alpha_0 - 1$ otherwise (cf. theorem 1).

3) To determine β^{**} note first that theorem 1 implies $\beta^{**}(x) \geq \alpha - \alpha_0$. Since this exponent also satisfies $\beta^{**}(x) \leq \liminf_{t \rightarrow x} \beta(t)$, as can be readily seen from its definition, and we have $\beta(x) = \alpha - \alpha_0$ for a dense set (the

irrational numbers if f is cuspidal and the non-cuspidal rationals otherwise) we conclude $\beta^{**}(x) = \alpha - \alpha_0$ for all x .

4) The case x_0 non-cuspidal has already been treated in the proof of theorem 1, part 3. Hence we may suppose that f is cuspidal at x_0 . We appeal again to theorem 4 but now we will need the explicit expression for the error term (*cf.* §4):

$$f_\alpha(x) = B|x - x_0|^{2\alpha}(x - x_0)^{-r} f_\alpha^\sigma(\sigma^{-1}x) + (10) + (12) + (13).$$

Terms (10) and (12) are everywhere differentiable, while term (13) can be differentiated at x_0 by lemma 15. Hence f_α is differentiable at x_0 if and only if the first summand is. Since f_α^σ is bounded, nonzero and periodic this will happen if and only if $2\alpha - r > 1$. The same must be true for the real and imaginary parts of f_α , since the image of f_α^σ is not contained in any one-dimensional subspace of \mathbb{C} .

Hence whenever $f'_\alpha(x_0)$ exists it is given by the sum of the derivatives of the terms (10) and (12) evaluated at x_0 (the other terms have vanishing derivative at x_0). Differentiating under the integral sign and integrating by parts one obtains the desired formula. \square

Proof of theorem 3. Let x_0 an irrational number. The pointwise Hölder exponent $\beta(x_0)$ is deduced by applying corollary 19 to the estimates of lemma 8 if f is a cusp form and of lemma 9 otherwise. The exponent $\beta^*(x_0)$ follows from lemma 20, while $\beta^{**}(x_0)$ was already determined in the proof of theorem 2, part 3. \square

7 Spectrum of singularities

In order to prove theorem 6 we will need some tools from diophantine analysis. More concretely we will need a refinement of the following classic theorem:

Theorem 21 (Jarník-Besicovitch). *Let $\tau \geq 2$. The Hausdorff dimension of the set*

$$A_\tau := \left\{ x : \left| x - \frac{p}{q} \right| \ll \frac{1}{q^\tau} \text{ for infinitely many rationals } \frac{p}{q} \right\} \quad (26)$$

is $2/\tau$. Moreover, if we denote by \mathcal{H}^t the t -dimensional outer Hausdorff measure,

$$\mathcal{H}^{2/\tau} \left(\bigcap_{\tau' < \tau} A_{\tau'} \right) > 0.$$

For the proof of theorem 21 when $\tau > 2$ we refer the reader to the book [9]: the first statement corresponds to proposition 10.3, while the second one is an immediate consequence of propositions 8.5, 8.6 and 10.4. In the limit case $\tau = 2$ we adopt the convention that $\bigcap_{\tau' < 2} A_{\tau'} = A_2$ (equality which is true anyway if one considers $A_{\tau'}$ to be defined for $1 < \tau' < 2$). The proof in this case follows from the fact that A_2 is the set of all irrational numbers (cf. theorem 5 of [10]).

Throughout this section it will be convenient to redefine “cusp” to mean an equivalence class of rationals modulo Γ (denoted by $\mathfrak{a}, \mathfrak{b}, \dots$). The theorem we need is the following, which takes into account that rational numbers are well distributed among the different cusps of Γ .

Theorem 22. *Let \mathfrak{a} be a cusp for Γ and $\tau \geq 2$. The Hausdorff dimension of the set*

$$A_\tau^\mathfrak{a} := \left\{ x : \left| x - \frac{p}{q} \right| \ll \frac{1}{q^\tau} \text{ for infinitely many rationals } \frac{p}{q} \in \mathfrak{a} \right\}$$

is $2/\tau$. Moreover, if we denote by \mathcal{H}^t the t -dimensional outer Hausdorff measure,

$$\mathcal{H}^{2/\tau} \left(\bigcap_{\tau' < \tau} A_{\tau'}^\mathfrak{a} \right) > 0.$$

Theorem 22 is a particular case of more general results about Fuchsian groups (cf. [23]). We provide here an elemental proof based on theorem 21.

Proof. Before beginning the proof, we note that we may assume without loss of generality that Γ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$. If this is not the case, we simply replace Γ with the biggest normal group it contains, *i.e.*, the intersection of all its conjugates.

Let γ be any matrix in $\mathrm{SL}_2(\mathbb{Z})$ and x an irrational number in $A_\tau^\mathfrak{a}$. We claim that if p/q is a rational number in a neighborhood of x and q' denotes the denominator of $\gamma(p/q)$ then $q' \ll q$. Indeed $q' = cp + dq$, and $p \ll q$ because $|p/q| \sim |x|$. From this together with the mean value theorem applied to $|\gamma(x) - \gamma(p/q)|$ we deduce that $\gamma(x) \in A_\tau^{\gamma(\mathfrak{a})}$. The argument can also be applied to γ^{-1} and therefore:

$$\gamma(A_\tau^\mathfrak{a}) = A_\tau^{\gamma(\mathfrak{a})}. \quad (27)$$

Note that the normality of Γ implies that $\gamma(\mathfrak{a})$ is again a cusp for the same group.

For any Lipschitz function φ with Lipschitz constant C and any set Ω we have

$$\mathcal{H}^t(\varphi(\Omega)) \leq C^t \mathcal{H}^t(\Omega). \quad (28)$$

This follows from the definition of Hausdorff outer measure. We want to apply this to prove that all the sets $A_\tau^\mathfrak{a}$ have roughly the same size when \mathfrak{a} ranges through the cusps of Γ , but the Möbius transformation γ is not Lipschitz in any neighborhood of its pole. This problem has a simple workaround. Let m be the width of the cusp ∞ and I any interval of length m not containing the pole of γ , and whose image $J = \gamma(I)$ is also of length m . Then from (27) we have

$$\begin{aligned} \gamma(A_\tau^\mathfrak{a} \cap I) &= A_\tau^{\gamma(\mathfrak{a})} \cap J \\ A_\tau^\mathfrak{a} + m &= A_\tau^\mathfrak{a}. \end{aligned}$$

Applying (28),

$$\mathcal{H}^t(A_\tau^{\gamma(\mathfrak{a})}) \ll \mathcal{H}^t(A_\tau^\mathfrak{a}).$$

The opposite inequality is also true and hence the Hausdorff dimension of the set $A_\tau^\mathfrak{a}$ must be independent of \mathfrak{a} . Since we also know by theorem 21 that $A_\tau = \bigcup_{\mathfrak{a}} A_\tau^\mathfrak{a}$ has dimension $2/\tau$, we conclude that all the $A_\tau^\mathfrak{a}$ must have exactly that dimension.

The very same argument applied to $\bigcap_{\tau' < \tau} A_{\tau'}^\mathfrak{a}$, instead of $A_\tau^\mathfrak{a}$ yields

$$\mathcal{H}^{2/\tau} \left(\bigcap_{\tau' < \tau} A_{\tau'}^{\gamma(\mathfrak{a})} \right) \ll \mathcal{H}^{2/\tau} \left(\bigcap_{\tau' < \tau} A_{\tau'}^\mathfrak{a} \right). \quad (29)$$

We claim the following is true:

$$\bigcup_{\mathfrak{a}} \bigcap_{\tau' < \tau} A_{\tau'}^\mathfrak{a} = \bigcap_{\tau' < \tau} A_{\tau'}. \quad (30)$$

From (29) and (30) we must have $\mathcal{H}^{2/\tau}(\bigcap_{\tau' < \tau} A_{\tau'}^\mathfrak{a}) > 0$, since otherwise we would contradict the second statement of theorem 21.

We prove (30) now. The inclusion \subset follows from $A_\tau^\mathfrak{a} \subset A_\tau$. For the opposite inclusion suppose x pertains to the set of the right hand side, *i.e.*, that for every $\tau' < \tau$ there is a constant $C_{\tau'} > 0$ such that

$$\left| x - \frac{p}{q} \right| \leq \frac{C_{\tau'}}{q^{\tau'}} \quad (31)$$

is satisfied for infinitely many rationals p/q . Let $\Xi_{\tau'}$ be the set of the cusps \mathfrak{a} for which (31) is satisfied for infinitely many rationals $p/q \in \mathfrak{a}$. This set is finite, nonempty and decreases with τ' . This implies that there exists some $\mathfrak{b} \in \bigcap_{\tau' < \tau} \Xi_{\tau'}$ and, therefore, $x \in \bigcap_{\tau' < \tau} A_{\tau'}^\mathfrak{b}$. \square

Corollary 23. *Let $2 \leq \tau \leq +\infty$. The Hausdorff dimension of the set $\{x : \tau_x = \tau\}$ is $2/\tau$.*

For the definition of τ_x see (6).

Proof. Assume $\tau > 2$ and let Ξ be the set of all the cusps at which f is not cuspidal. We have the identity

$$\{x : \tau_x = \tau\} = \bigcap_{\tau' < \tau} \bigcup_{\mathfrak{a} \in \Xi} A_{\tau'}^{\mathfrak{a}} \setminus \bigcup_{\tau' > \tau} \bigcup_{\mathfrak{a} \in \Xi} A_{\tau'}^{\mathfrak{a}}.$$

By theorem 22 the set on the right hand side has Hausdorff dimension at most $2/\tau$. On the other hand from the same theorem one deduces that for $\tau < +\infty$ we have

$$\mathcal{H}^{2/\tau} \left(\bigcap_{\tau' < \tau} \bigcup_{\mathfrak{a} \in \Xi} A_{\tau'}^{\mathfrak{a}} \right) > 0, \quad \mathcal{H}^{2/\tau} \left(\bigcup_{\tau' > \tau} \bigcup_{\mathfrak{a} \in \Xi} A_{\tau'}^{\mathfrak{a}} \right) = 0.$$

This implies the other inequality for the Hausdorff dimension.

The case $\tau = 2$ follows from the fact that $\tau_x \geq 2$ for every irrational number x (cf. [20]). \square

Proof of theorem 6. The set $\{x : \beta(x) = \delta\}$ is completely determined by theorems 2 and 3. Its Hausdorff dimension in the case of cuspidal f is immediate, while if f is noncuspidal it follows from corollary 23. \square

8 Riemann's example

In this section we employ the developed machinery to explain some aspects of the graph of Riemann's example (1), plotted in figure 1. A similar but more detailed exposition is given by Duistermaat in [7]. Our analysis, however, is readily applicable to any other modular form.

Riemann's example φ satisfies $2\varphi(x) = \Im \theta_1(x)$, where θ corresponds to Jacobi's theta function $\theta(z) = \sum_{n \in \mathbb{Z}} e^{n^2 \pi i z}$. This is a modular form of weight $1/2$ for the group Γ_θ , consisting of all matrices in $\mathrm{SL}_2(\mathbb{Z})$ of the form $\begin{pmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{pmatrix}$ or $\begin{pmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{pmatrix}$. The Γ_θ -orbit of 0 corresponds to ∞ together with all the rationals p/q with either p even and q odd, or p odd and q even. All the remaining rationals (p/q with both p and q odd) constitute the Γ_θ -orbit of 1. The modular form θ is cuspidal at 1 but not at 0 and the associated multipliers μ_γ are always 8th roots of unity. The proofs of these statements can be consulted in [7].

Note that the previous facts are enough to recover Hardy's and Gerver's theorems and determine the Hölder exponents of φ at every point. Its spectrum of singularities, first obtained by Jaffard in [17], also follows from theorem 6.

Jacobi's function θ is classically denoted ϑ_3 , as it has two companions which are also modular forms of weight $1/2$ for conjugated groups of Γ_θ :

$$\tilde{\theta}(z) = \vartheta_2(z) = \sum_{n \in \mathbb{Z}} e^{(n+\frac{1}{2})^2 \pi i z} \quad \text{and} \quad \theta(z+1) = \vartheta_4(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{n^2 \pi i z}.$$

The nomenclature $\tilde{\theta}$ is not standard but we employ it here as a convenient way to avoid problems with subscripts.

Given any matrix $\sigma \in \text{SL}_2(\mathbb{Z})$ the modular form θ^σ is either a multiple of $\vartheta_2 = \tilde{\theta}$, $\vartheta_3 = \theta$ or $\vartheta_4(z) = \theta(z+1)$, the constant being an 8th root of unity (see theorem 7.1.2 of [21]). Since θ^σ is cuspidal at ∞ if and only if $\theta(\sigma(\infty)) = 0$, one concludes that:

$$\theta^\sigma(z) = \begin{cases} C\theta(z) \text{ or } C\theta(z+1) & \text{if } \sigma(\infty) \in \Gamma_\theta \cdot 0 \\ C\tilde{\theta}(z) & \text{if } \sigma(\infty) \in \Gamma_\theta \cdot 1. \end{cases}$$

We now apply theorem 4 with $\alpha = 1$, $r = 1/2$, to study the behavior of $\varphi = \frac{1}{2}\Im\theta_1$ in the neighborhood of a given rational point x_0 . The resulting expansion around x_0 is of the form:

$$\varphi(x) = \Im \left[C\sqrt{x-x_0} \right] + \Im \left[C'(x-x_0)^{3/2} f_1(\sigma^{-1}x + \tau) \right] + h(x).$$

The constant C is nonzero if and only if $x_0 \in \Gamma \cdot 0$, and in this case $f = \theta$. Otherwise $f = \tilde{\theta}$. The constant C' is always nonzero, and both constants have the argument of an 8th root of unity. Finally, τ is either 0 or 1.

Some deductions are immediate. The first one being that φ has singularities of square root type at every rational of the form odd/even or even/odd (either at one side or both sides of the rational), while it is differentiable at every rational of the the form odd/odd. The second one is that at either side of any rational number φ mimics the graph of some periodic function $\Im C' f_1$. Note that as σ^{-1} has a simple pole at x_0 , this pattern repeats indefinitely towards the rational, with its amplitude decreasing as a $3/2$ power of the remaining distance and its frequency roughly proportional to $|x-x_0|^{-1}$. See figure 2 for some examples of this behavior, where the square root singularities are also clearly visible.

Since the argument of C' is an integer multiple of $\pi/4$ we also deduce that $\Im C' f_1$ is either $\Re f_1$, $\Im f_1$ or $\Re f_1 + \Im f_1$, or the mirror image of one

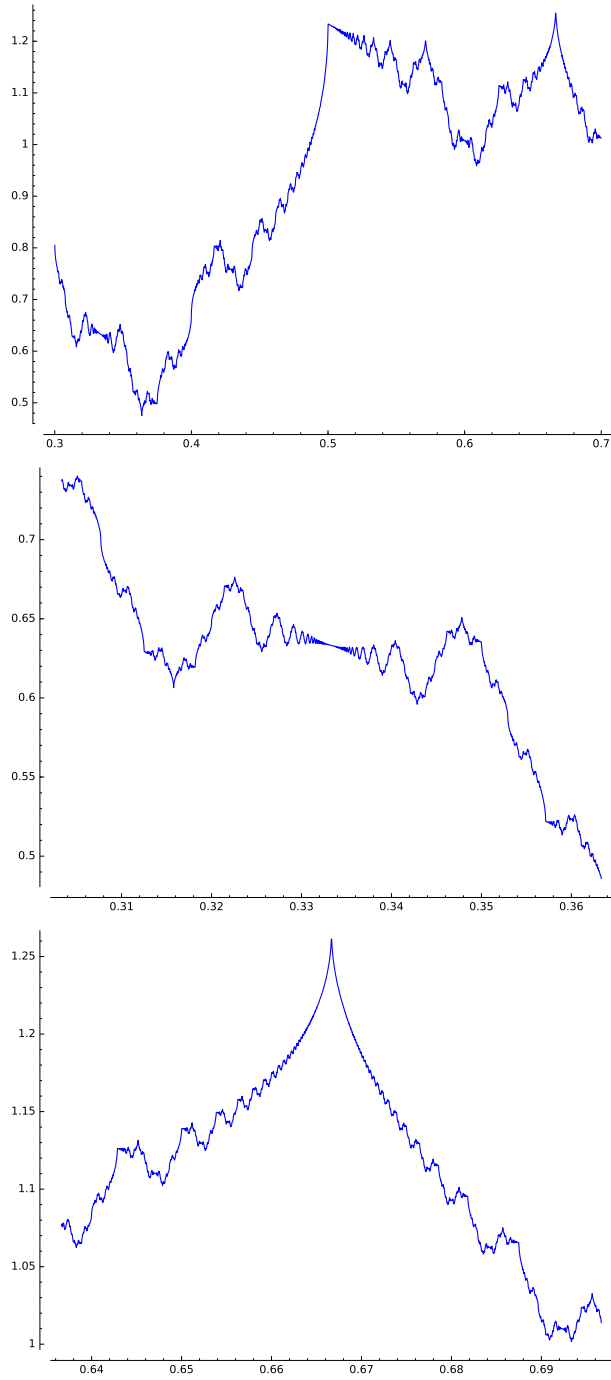


Figure 2: Detail of φ near $1/2$, $1/3$ and $2/3$, respectively.

of these functions, *i.e.*, the result of performing the change of variables $x \mapsto -x$ either in the domain, in the codomain or both. The situation is even simpler when $f = \tilde{\theta}$, as all these functions are then translates of each other (*cf.* theorem 7.1.2 of [21]). Hence the graph of $\Im C' f_1$ corresponds, up to symmetries, to one of the four genuinely distinct patterns that appear in figure 3. Note that in figure 2 all four patterns appear.

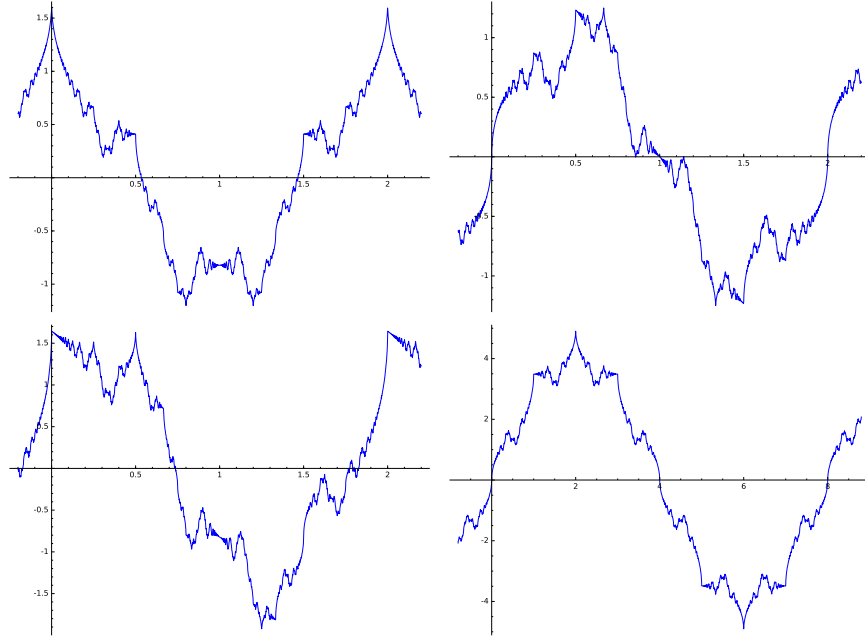


Figure 3: Graphs of $\Re\theta_1$ (top-left), $\Im\theta_1$ (top-right), $\Re\theta_1 + \Im\theta_1$ (bottom-left) and $\Im\tilde{\theta}_1$ (bottom-right).

The previous discussion shows that the graph of φ is a fractal in the sense that a slightly deformed version of itself appears at the sides of some rational numbers (for example on the left of $1/2$, see figure 2). But it is also a fractal in the sense that it exhibits approximate self-similarity around every rational number, as can be seen from theorem 4 by choosing a parabolic matrix σ in Γ fixing a certain rational. The same argument can be employed to show local self-similarity around quadratic surds, as each of these points is fixed by some hyperbolic transformation in Γ_θ (see [7] for details). We remark that both statements are also true for any other finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. Indeed, if σ is parabolic (hyperbolic) and fixes a certain point, then some power σ^n lies in Γ , is also parabolic (hyperbolic) and fixes the same point.

9 Cusp forms for $\Gamma_0(N)$

Fix an arbitrary integer $N \geq 1$ and let f be a cusp form of integer weight r for the group $\Gamma_0(N)$ and trivial multiplier system. Note that r must necessarily be even. For any $\alpha > r/2$ the function f_α is well-defined and we may consider $g = \Re f_\alpha$ or $\Im f_\alpha$. Since the factor $B|x - x_0|^{2\alpha-r}$ in theorem 4 is always positive, the analysis of §8 shows that the graph of g “repeats itself” near rational points in the orbit $\Gamma_0(N) \cdot \infty$, in the sense that we should expect oscillations of the form $(x - x_0)^{2\alpha-r} g(\sigma^{-1}x)$ near these points. What is less obvious is that this might also happen for other rationals not lying in the orbit of ∞ , for example if we are able to find some matrix $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with prescribed $\sigma(\infty)$ and for which the form $f^\sigma = f|_\sigma$ equals Cf for some real constant C . A particular case of this is shown in figure 4.

A good place to look for such a σ is in the normalizer of $\Gamma_0(N)$, as this automatically implies that $f|_\sigma$ is also a modular form for the same group and trivial multiplier system. In this section we give some sufficient conditions that guarantee the existence of such σ for every rational number, and study some particular cases where the pattern repeating around some rational numbers bears little resemblance with the global graph of g .

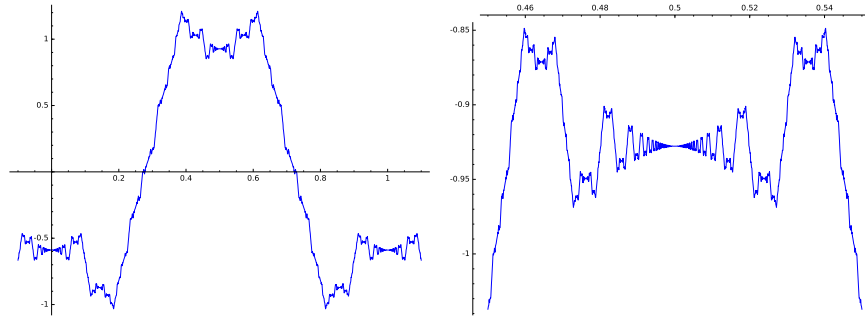


Figure 4: Left: Plot of $-\Re f_{9/5}$, where f is the newform on $\Gamma_0(14)$. Right: detail of $\Re f_{9/5}$ at $1/2$. This rational is not in $\Gamma_0(14) \cdot \infty$, but the matrix $\sigma = \begin{pmatrix} 7 & 3 \\ 14 & 7 \end{pmatrix}$ satisfies $\sigma(\infty) = 1/2$ and $f|_\sigma = -f$.

Some notation first. For any two integers n and m we denote by (n, m) its greatest common divisor, and for every prime p we denote by $[n]_p$ the largest power of p dividing n . For every divisor $Q \mid N$ satisfying $(Q, N/Q) = 1$ we define the matrix

$$\omega_Q := \begin{pmatrix} Qx & y \\ Nz & Qw \end{pmatrix}, \quad x, y, z, w \in \mathbb{Z}, \quad \det \omega_Q = Q,$$

which is unique up to left and right multiplication by elements of $\Gamma_0(N)$. The matrices ω_Q are called Atkin-Lehner involutions and satisfy $Q^{-1}\omega_Q^2 \in \Gamma_0(N)$ and $\omega_Q\omega_{Q'} = \text{some } \omega_{QQ'}$ whenever $(Q, Q') = 1$. For the sake of clarity we also set $\omega_p := \omega_{[N]_p}$ for each prime $p \mid N$. Finally for any integer $n > 0$ we consider the matrix

$$S_n := \begin{pmatrix} 1 & 1/n \\ 0 & 1 \end{pmatrix},$$

which corresponds to a translation by $1/n$.

A theorem of Atkin and Lehner stated without proof in [2] assures that when N is not divisible by 4 nor 9 the normalizer of $\Gamma_0(N)$ is generated by $\Gamma_0(N)$ and the Atkin-Lehner involutions ω_p for primes $p \mid N$. When N is divisible by 4 or by 9 one has to include some extra generators: S_2 if $[N]_2 = 4$ or 8, S_4 if $[N]_2 = 16$ or 32 and S_8 if $64 \mid N$; and S_3 if $9 \mid N$. Note that we are thinking of the normalizer of $\Gamma_0(N)$ as a group of linear fractional transformations, as otherwise one also needs to include any real multiple of the previous generators. This theorem also provides the structure of the quotient between these two groups (which we do not need), although this part seems to have some mistakes and a corrected version is proved by Bars in [3].

Asai observed in [1] that the Atkin-Lehner involutions act transitively on \mathbb{Q} if and only if N is square-free. The following proposition is a generalization of this fact.

Proposition 24. *The normalizer of $\Gamma_0(N)$ acts transitively on \mathbb{Q} if and only if $N = 2^a 3^b N'$ for some $a < 8$, $b < 4$ and a square-free integer N' not divisible by 2 nor by 3.*

Proof. Assume first that N is of the prescribed form. We begin by proving that every rational u/v is related by the Atkin-Lehner involutions to some other rational u'/v' with $N' \mid v'$. All the integers u, u', \dots and v, v', \dots involved are assumed to satisfy $(u, v) = 1$, $(u', v') = 1$, etc.

Write $N' = p_1 \cdots p_n$ for distinct primes p_1, \dots, p_n . We may assume upon reordering of the p_i that $p_1 \cdots p_m \mid v$ and $p_i \nmid v$ for $m < i \leq n$. Choosing $Q = 2^a 3^b p_{m+1} \cdots p_n$ we have

$$u'/v' = \omega_Q(u/v) = \frac{Qxu + yv}{N \left(zu + w \frac{v}{N/Q} \right)}.$$

The numerator of the right hand side is not divisible by any of the p_i as a consequence of the determinant condition imposed on ω_Q and therefore $N' \mid v'$.

Next we prove that the rational u'/v' is related to some u''/v'' with $2^a N' \mid v''$. In order to do this we employ ω_2 and the translations S_2, S_4 or S_8 that lie in the normalizer. It is easy to check that all these transformations preserve $(v', 3^a N')$ and therefore they may be applied in any order without worrying about the condition $N' \mid v''$. Let $2^s = [v']_2$ and assume that $s < a$, since otherwise we are finished. It is also easy to check that if $u''/v'' = \omega_2(u'/v')$ then $[v'']_2 = 2^{a-s}$. This means that applying ω_2 if necessary we may assume $s \leq [a/2]$. We now apply repeatedly S_2, S_4 or S_8 to arrive to a rational with $s = 0$, and the image of this rational by ω_2 satisfies $s \geq a$.

The same argument can now be applied *mutatis mutandis* to relate the rational u''/v'' to some u'''/v''' with $N \mid v'''$, *i.e.*, lying in the orbit $\Gamma_0(N) \cdot \infty$. This finishes the proof of the direct implication.

To prove that the normalizer action is not transitive when N is not of the prescribed form it suffices to show a subset of \mathbb{Q} invariant under this action. Suppose first that for some prime $p \neq 2, 3$ we have $p^2 \mid N$ and $p^c = [N]_p$. Then one such set is that of the rational numbers u/v with $[v]_p = p^s$ with $0 < s < c$. The invariance of this set follows from the following facts: the translations and the Atkin-Lehner involutions ω_Q with $p \nmid Q$ leave $[v]_p$ invariant, while $[v']_p = p^{c-s}$ for $u'/v' = \omega_Q(u/v)$ with $p \mid Q$.

The remaining cases are $2^8 \mid N$ or $3^4 \mid N$. If $2^8 \mid N$ then $a \geq 8$ and one such set is that of the rational numbers u/v with $[v]_2 = 2^{a/2}$ if a is even and $[v]_2 = 2^{[a/2]}$ or $[v]_2 = 2^{[a/2]+1}$ if a is odd. An analogous set works when $3^4 \mid N$. \square

Suppose now that f is a newform. Atkin and Lehner proved in [2] that $f|_{\omega_p} = \pm f$ for every prime $p \mid N$. In the same paper they also prove that when $4 \mid N$ all the even coefficients of f vanish, and therefore $f|_{S_2} = -f$. For the rest of translations considered above (S_4, S_8 and S_3), however, it is not generally true that $f|_{S_n} = Cf$ for some real constant C , not even when S_n belongs to the normalizer of $\Gamma_0(N)$. In fact $f|_{S_n}$ might not be a newform at all, as it is shown below. These translations are necessary to make the action of the normalizer transitive on \mathbb{Q} when $2^4 \mid N$ or $9 \mid N$ (*cf.* proof of proposition 24), and therefore chances are that there will be some rationals p/q for which no matrix σ satisfies $\sigma(\infty) = p/q$ and $f|_{\sigma} = Cf$.

There is an exception to this: when the space of cuspidal forms has dimension 1 (*i.e.* there are no oldforms and every cusp form is a multiple of f). In this case all the matrices in the normalizer of $\Gamma_0(N)$ commute under the slash operator, and therefore if η is any of these matrices we must have $f|_{\eta} = f|_{\omega_Q S} = \pm f|_S$ for some $Q \mid N$ and some translation S . Note that as long as the action of the normalizer on \mathbb{Q} is transitive we are free to

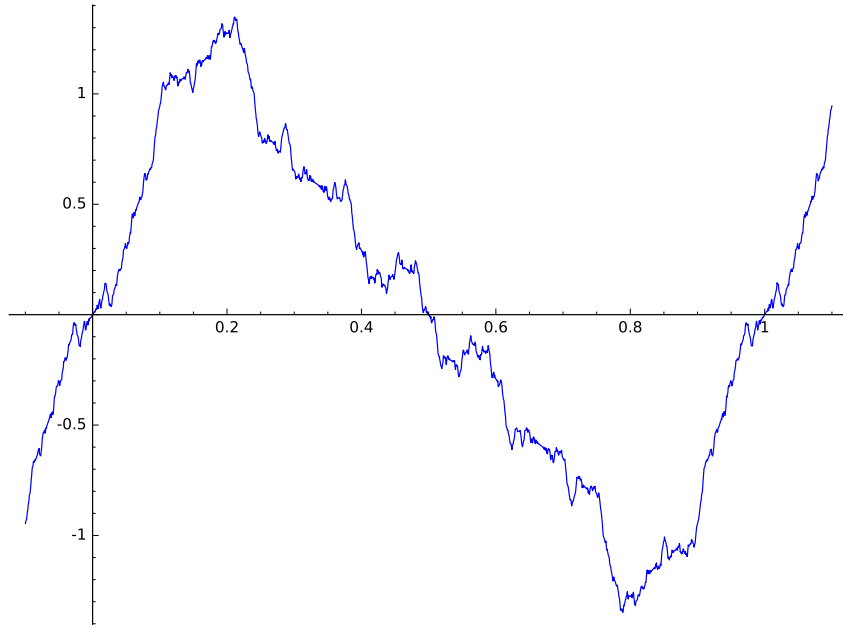


Figure 5: Plot of $\Im f_{7/4}$ where f is the newform on $\Gamma_0(45)$.

choose $\eta(\infty)$. The matrix $\sigma = \eta S^{-1}$ now lies in the normalizer and satisfies $\sigma(\infty) = \eta(\infty)$ and $f|_\sigma = \pm f$.

We conclude that the graph of $\Re f_\alpha$ or $\Im f_\alpha$ repeats itself around every rational number as long as $N = 2^a N'$ with $a < 4$ and N' odd and square-free, or if the space of cusp forms on $\Gamma_0(N)$ has dimension 1 and $N = 2^a 3^b N'$ with $a < 8$, $b < 4$ and N' square-free and not divisible by 2 nor 3.

We now give some examples for which a different pattern appears around some rational numbers. These are of weight 2 and therefore associated to elliptic curves over \mathbb{Q} (*cf.* §1). By direct examination of the table of newforms found at [19] we see that the lowest value of N for which neither of the previous conditions is satisfied is $N = 45$, as the associated space of cusp forms happens to be of dimension 3, containing an oldclass generated by the newform on $\Gamma_0(15)$. Denote by f the newform on $\Gamma_0(45)$ and by g the one on $\Gamma_0(15)$. These correspond to the isogeny classes of the elliptic curves

$$y^2 + xy = x^3 - x^2 - 5 \quad \text{and} \quad y^2 + xy + y = x^3 + x^2,$$

respectively. The matrix $\sigma = S_3 \omega_{45}$, where ω_{45} is the Atkin-Lehner involution determined by $x = w = 0$, $y = 1$ and $z = -1$, lies in the normalizer of

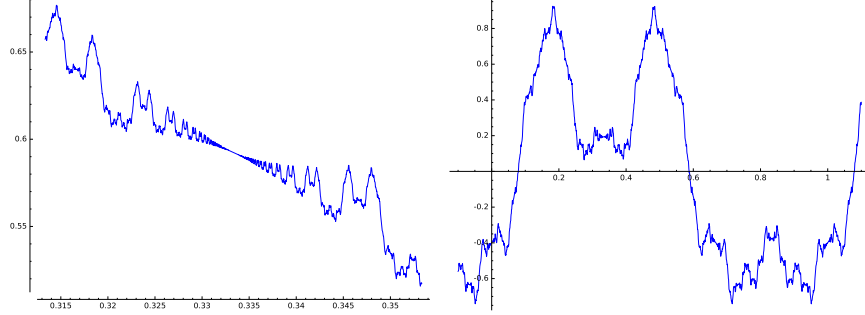


Figure 6: Left: Detail of $\Im f_{7/4}$ around $1/3$ where f is the newform on $\Gamma_0(45)$. Right: Graph of the imaginary part of the right hand side of (32).

$\Gamma_0(45)$ and sends ∞ to $1/3$. The function $f|_\sigma$ is therefore again a modular form for $\Gamma_0(45)$, and in fact it has the following decomposition:

$$f|_\sigma(z) = \frac{1}{2}f(z) - i\frac{1}{2\sqrt{3}}g(z) - i\frac{3\sqrt{3}}{2}g(3z).$$

To obtain the coefficients one first decomposes $f|_{S_3}$ by directly comparing coefficients, and then applies $|\omega_{45}$. The Atkin-Lehner eigenvalues are tabulated in [19], and the action of this operator on oldforms is described by lemma 26 of [2]. As an immediate consequence

$$f_\alpha^\sigma(x) = \frac{1}{2}f_\alpha(x) - \frac{i}{2\sqrt{3}}g_\alpha(x) - \frac{i}{2 \cdot 3^{\alpha-3/2}}g_\alpha(3x). \quad (32)$$

In figure 5 we have plotted $\Im f_{7/4}$, while in figure 6 the reader can compare the imaginary part of the right hand side of (32) for $\alpha = 7/4$ with the pattern that repeats near $\sigma(\infty) = 1/3$ for $\Im f_{7/4}$.

The lowest value of N for which the normalizer is not transitive on \mathbb{Q} and there is some nonzero newform is $N = 49$. This newform is associated to the isogeny class of the curve

$$y^2 + xy = x^3 - x^2 - 2x - 1.$$

The cusp $1/7$ is not related to ∞ , not even by the normalizer, and in figure 7 the reader can appreciate that the global graph and the pattern appearing around this point have a different aspect.

To finish this section we note that although Jacobi's theta function θ was presented in §8 as modular for the group Γ_θ , the closely related function $\theta(2z)$ is a modular form for $\Gamma_0(4)$. The normalizer of $\Gamma_0(4)$ does act

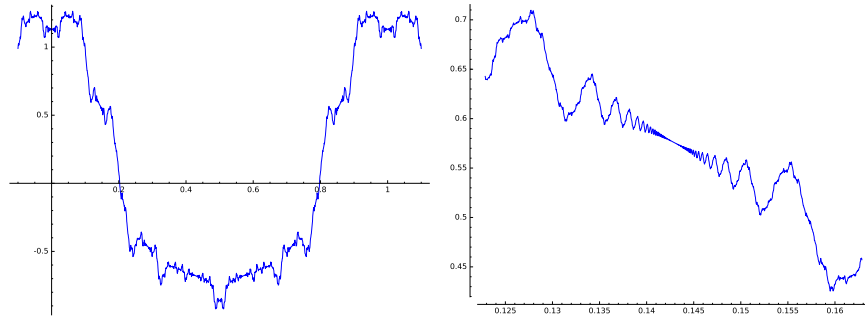


Figure 7: Left: Plot of $\Re f_{7/4}$ where f is the newform on $\Gamma_0(49)$. Right: Detail around $1/7$.

transitively on \mathbb{Q} . The reason four possible patterns can appear around rational points is that in this case the multiplier system is not trivial, and therefore it might change when one applies the slash operator $|\sigma$.

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The graphics included in the article have been plotted using *Sage*, and the same software system has been used to compute the Fourier coefficients of newforms. The partial sums were calculated using simple C++ programs.

References

- [1] T. Asai. *On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution*. J. Math. Soc. Japan, 28(1):48–60, 1976.
- [2] A. O. L. Atkin, J. Lehner. *Hecke operators on $\Gamma_0(m)$* . Math. Ann., 185:134–160, 1970.
- [3] F. Bars. *The group structure of the normalizer of $\Gamma_0(N)$* . arXiv:math/0701636v1.

- [4] P. I. Butzer, E. I. Stark. “*Riemann’s example*” of a continuous nondifferentiable function in the light of two letters (1865) of Christoffel to Prym. *Bull. Soc. Math. Belg.*, 38:45–73, 1986.
- [5] F. Chamizo. *Automorphic Forms and Differentiability Properties*. *Trans. Amer. Math. Soc.*, 356(5):1909–1935 (electronic), 2004.
- [6] F. Chamizo, I. Petrykiewicz, S. Ruiz-Cabello. *The Hölder exponent of some Fourier series*. arXiv:1504.04998v1, 2015. Preprint.
- [7] J. J. Duistermaat. *Selfsimilarity of “Riemann’s Nondifferentiable Function”*. *Nieuw Arch. Wisk.*, 9(3):303–337, 1991.
- [8] M. Eichler. *Eine Verallgemeinerung der Abelschen Integrale*. *Math. Z.* 67:267–298, 1957.
- [9] K. Falconer. *Fractal geometry*. John Wiley and sons, 2003.
- [10] J. R. Ford. *Fractions*. *Amer. Math. Monthly*, 45(9):586–601, 1938.
- [11] J. Gerver. *The differentiability of the Riemann function at certain rational multiples of π* . *Amer. J. Math.*, 92:33–55, 1970.
- [12] J. Gerver. *More on the differentiability of the Riemann function*. *Am. J. Math.*, 93(1):33–41, 1971.
- [13] G. H. Hardy. *Weierstrass’s nondifferentiable function*. *Trans. Amer. Math. Soc.*, 17(3):301–325, 1916.
- [14] M. Holschneider, Ph. Tchamitchian. *Pointwise analysis of Riemann’s “nondifferentiable” function*. *Invent. Math.*, 105(1):157–175, 1991.
- [15] R. Iorio, V. Iorio. *Fourier Analysis and Partial Differential Equations*. Vol. 70 of *Cambridge Stud. Adv. Math.*. Cambridge Univ. Press, 2001.
- [16] H. Iwaniec. *Topics in Classical Automorphic Forms*. Vol. 17 of *Graduate Studies in Mathematics*, Amer. Math. Soc., 1997.
- [17] S. Jaffard. *The spectrum of singularities of Riemann’s function*. *Rev. Mat. Iberoamericana*, 12(2):441–460, 1996.
- [18] S. Jaffard. *Local behaviour of Riemann’s function*. In *Harmonic analysis and operator theory (Caracas, 1994)*, vol. 189 of *Contemp. Math.*, pp 287–307. Amer. Math. Soc, 1995.
- [19] The LMFDB Collaboration. *The L-functions and Modular Forms Database*. <http://www.lmfdb.org>, 2013. [Online; accessed 4 March 2016].
- [20] S. J. Patterson. *Diophantine approximation in Fuchsian groups*. *Phil. Trans. R. Soc. Lond. A*, 262:527–563, 1976.
- [21] R. A. Rankin. *Modular forms and functions*. Cambridge Univ. Press, 1977.

- [22] S. Seuret, J. L. Véhel. *The local Hölder function of a continuous function*. Appl. Comput. Harmon. Anal., 13(3):263–276, 2002.
- [23] S. L. Velani. *Diophantine approximation and Hausdorff dimension in Fuchsian groups*. Math. Proc. Cam. Phil. Soc., 113:343–354, 1993.
- [24] K. Weierstrass. *Über continuierliche Functionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten differentialquotienten besitzen*. In *Mathematische Werke II*, pp 71-74. Königl. Akad. Wiss., 1872.
- [25] E. T. Whittaker, G. N. Watson. *A course in modern analysis*. Cambridge Univ. Press, 1915.
- [26] A. Zygmund. *Trigonometric series*. Vol I, II. Cambridge Univ. Press, 2002.